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On the Instanton Contributions to the Masses and Couplings of E_6 Singlets

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ABSTRACT

We consider the gauge neutral matter in the low–energy effective action for string theory compactification on a Calabi–Yau manifold with $(2,2)$ world–sheet supersymmetry. At the classical level these states (the $\mathbf{1}$'s of E_6) correspond to the cohomology group $H^1(\mathcal{M}, \mathrm{End}\, T)$. We examine the first order contribution of instantons to the mass matrix of these particles. In principle, these corrections depend on the Kähler parameters t_i through factors of the form $e^{2\pi i t_i}$ and also depend on the complex structure parameters. For simplicity we consider in greatest detail the quintic threefold $\mathbb{P}_4[5]$. It follows on general grounds that the total mass is often, and perhaps always, zero. The contribution of individual instantons is however nonzero and the contribution of a given instanton may develop poles associated with instantons coalescing for certain values of the complex structure. This can happen when the underlying Calabi–Yau manifold is smooth. Hence these poles must cancel between the coalescing instantons in order that the superpotential be finite. We examine also the Yukawa couplings involving neutral matter $\mathbf{1}^3$ and neutral and charged fields $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$, which have been little investigated even though they are of phenomenological interest. We study the general conditions under which these couplings vanish classically. We also calculate the first–order world–sheet instanton correction to these couplings and argue that these also vanish.

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Contents

1. Introduction

2. Review of What is Known Classically

    2.1 Deformations of the complex structure and of the tangent bundle

    2.2 The number of singlets

    2.3 Integrability and the vanishing of the $\mathbf{1}^3$ *coupling*

    2.4 The vanishing of the $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ *coupling*

3. Preliminaries Concerning Instanton Contributions to Correlation Functions

    3.1 The zero modes

    3.2 Vertex operators and powers of g

    3.3 Decomposition of singlets along a line

4. Masses Generated by Instantons

    4.1 The mass matrix

    4.2 Other splitting types

5. Examples of the Effects of Lines in $\mathbb{P}_4[5]$

    5.1 General facts

    5.2 An $\mathcal{O}(0) \oplus \mathcal{O}(-2)$ *line as the limit of* $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ *lines*

    5.3 The singlet mass matrix is probably always zero

6. The Yukawa Couplings

    6.1 The couplings in terms of zero modes

    6.2 The form of the Yukawa couplings

7. Conclusion

    7.1 Summary and discussion

    7.2 Open questions

A. Remarks on the Determinants and Resolution of an Apparent Paradox

1. Introduction

Calabi–Yau compactifications correspond to string theory vacua with $(2,2)$ world–sheet supersymmetry. In these theories, there is now a good understanding of many aspects of the low–energy effective theories that correspond to these vacua. The families ($\mathbf{27}$'s) and anti-families ($\overline{\mathbf{27}}$'s) of E_6 are in one–one correspondence with the complex structure and Kähler class parameters of the manifold. The geometry of the parameter spaces is coming to be understood, and, in virtue of mirror symmetry, the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ Yukawa couplings may be calculated exactly in the sigma model. These Yukawa couplings may be expressed, equivalently, in terms of the periods of the holomorphic three–form over the manifold [1,2] or in terms of a convergent instanton expansion. There are, however, other parameters in these models [3] which correspond to E_6 singlets, $\mathbf{1}$. Geometrically, these parameters correspond to deformations of the tangent bundle of the Calabi–Yau manifold. The infinitesimal deformations of the tangent bundle correspond to the cohomology group $H^1(\mathcal{M}, \mathrm{End}\,\mathcal{T})$, which is generically nontrivial. Unfortunately, these parameters are of a more recondite character than is the case for the $\mathbf{27}$'s and $\overline{\mathbf{27}}$'s and, until recently [4], little was known about whether they acquire mass by non–perturbative corrections or about the Yukawa couplings, the $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ and $\mathbf{1}^3$, into which they enter. These couplings are of phenomenological interest [3,5], and our lack of understanding constitutes a considerable barrier to model building. An improved understanding of these parameters would also aid in the exploration of the largely mysterious class of $(0,2)$ vacua, which are potentially of considerable phenomenological importance. Quite apart from any phenomenological considerations, the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ Yukawa couplings are of interest because they reflect deep geometrical properties of the Calabi–Yau manifold and of its moduli space and it seems likely that the same should be true of the couplings that involve the singlets.

In the absence of any insight into the geometry of the parameter space corresponding to the singlets we focus, in this article, on the much more modest goal of studying the instanton contribution to the mass matrix of the singlets and the $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ and $\mathbf{1}^3$ Yukawa couplings following the techniques of early papers on the instanton contributions to the $\mathbf{27}^3$ and $\overline{\mathbf{27}}^3$ couplings [6,7]. To do the calculations, we follow [7] by writing down the path integral corresponding to the Yukawa coupling, which involves an insertion of three appropriate vertex operators:

$$y_{ijk} = \frac{\int \mathcal{D}[x]\, V_i V_j V_k e^{-S}}{\int \mathcal{D}[x]\, e^{-S}} \quad . \tag{1.1}$$

The integral breaks up into a sum over saddle points of different instanton degree; these correspond to the topologically different ways the world–sheet can be mapped into the

target space. This leads to an expansion:

$$\int \mathcal{D}[x]\, V_i V_j V_k e^{-S} = e^{-S_{free}|_0} \int \mathcal{D}[\tilde{x}]\, V_i V_j V_k e^{-S_{int}}\Big|_0 + \; e^{-S_{free}|_1} \int \mathcal{D}[\tilde{x}]\, V_i V_j V_k e^{-S_{int}}\Big|_1 \quad + \quad \ldots \tag{1.2}$$

The integrals that remain are over the modular group of the world–sheet and the zero modes of the instanton.

The mass matrix arises from an **R**.**1**.**1** coupling, where **R** denotes the "dilaton" (the scale of the internal manifold) and so is, in reality, a Yukawa coupling also. There is also a **C**.**1**.**1** coupling, where **C** is the vertex operator for a complex structure modulus, that is related by mirror symmetry to **R**.**1**.**1** of the mirror manifold. Logically[1] the study of the mass matrix precedes the study of the couplings $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ and $\mathbf{1}^3$, since those singlets that acquire mass are absent from the low energy theory and the couplings $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ and $\mathbf{1}^3$ into which they enter are irrelevant. Furthermore we can only calculate on-shell quantities in string theory. The superpotential couplings, that is zero-momentum 2– and 3–point couplings of the chiral fields, are on-shell *provided* the fields are massless. If we find that a certain singlet is massive (appears quadratically in the superpotential), then the string calculations of its 3– and higher–point couplings are off-shell, and hence ambiguous.

This investigation began with a computation of the instanton corrections to the $\mathbf{1}^3$ and $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ couplings. The resulting expressions fail to vanish owing to certain δ-function contributions that arise on integrating by parts. The resulting couplings are ambiguous in that they are ill-defined in BRST cohomology. It appeared that there were two possible resolutions to this. On the one hand the singlet could acquire mass through instanton corrections in which case one would expect the couplings to be ill-defined in BRST cohomology and they would also be irrelevant since they would not be couplings between massless particles. A second possibility is that the ambiguous contributions should be removed by a proper accounting for contact terms in the conformal field theory.

In studying the the mass matrix for the **1**'s we find that the individual instantons contribute amounts that are generically non-zero. A general argument following [4], however implies that these must cancel in the sum. Moreover all contributions to the superpotential that involve the **1**'s should vanish. The consequence for the couplings is that the ambiguous contributions to the Yukawa couplings should be canceled by contact terms. In fact once we know that the masses vanish, the particular form of our expressions is such that the only way to ascribe unambiguous meaning to them is that they should indeed vanish.

We find the individual contributions of the instantons to be rather complicated. For example each instanton, L, makes a contribution to the mass matrix that depends on

[1] We are grateful to E. Witten for stressing this point to us.

the complex structure and Kähler parameters of the manifold *and* on the particular instanton L. For the $\overline{\mathbf{27}}^3$ coupling, by contrast, the coupling depends only on the Kähler parameters of the manifold and, in each degree, each instanton contributes equally. That is, it is sufficient to know the number of instantons of each degree, and it is not necessary to know the location of each instanton. For the singlet couplings, however, much more detailed information is necessary. As the complex structure parameters of the manifold are varied, the instantons move in the manifold; though if properly counted, their number remains fixed. Thus a zero total mass is achieved by very complicated cancellations between the contributions of the different instantons. This situation is of potential interest to mathematicians; it is possible to ascribe a matrix, computed from $H^1(\mathcal{M}, \mathrm{End}\,\mathcal{T})$, to each instanton in such a way that the sum over all instantons vanish. It is clear that this persists to all orders in the instanton expansion.

The layout of this paper is the following. In Section 2 we review what is known about the classical values of the $\mathbf{1}^3$ couplings and we show that the $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ couplings vanish whenever the singlet corresponds to a polynomial deformation of the tangent bundle. In Section 3 we gather together the basic elements that we need to calculate the instanton corrections to the couplings. These are the zero modes of the fields about the instanton and the form of the vertex operators that we will need. A basic technique is the decomposition of tensors into components that are sections of line bundles over the instanton and the application of the Bott–Borel–Weil theorem which gives the cohomology of forms that take values in these bundles. We then discuss, in Section 4, the form of the mass matrix for the singlets. In particular we consider the mass matrix due to the simplest type of instanton, which has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. This is in some sense the generic case but other cases are possible and indeed arise as the parameters of the Calabi–Yau manifold are varied. It can happen, for example, that for special values of the parameters, two (or more) instantons coincide and the normal bundle of each degenerates to $\mathcal{O}(0) \oplus \mathcal{O}(-2)$. In this situation the contribution of each line to the mass matrix has a pole; though the poles cancel between the coalescing lines. We examine this situation in Section 5 in the context of the manifold $\mathbb{P}_4[5]$ and extend the argument of [4] to show that the total mass and all the couplings involving $\mathbf{1}$ vanish for a wide variety of situations. We turn, in Section 6, to a detailed computation of the first order instanton corrections to the Yukawa couplings $\mathbf{1}^3$ and $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ and a discussion of the resulting expression. In Section 7 we discuss our results and list a number of open questions. Finally, an appendix deals with the determinants associated to expansion about the instanton and issues related to the zero modes about multiple instantons that result from the coalescence of isolated instantons.

2. **Review of What is Known Classically**

2.1. Deformations of the complex structure and of the tangent bundle

At least some (and in favourable circumstances all) deformations of the tangent bundle $\mathcal{T}$ on the manifold $\mathcal{M}$ may be described using the Kodaira-Spencer deformation theory [8] as presented, for example, in Refs. [9,10]. A vector tangent to a projective space may be understood as a differential operator $v^A \frac{\partial}{\partial z^A}$ that acts on functions homogeneous of degree zero. Since

$$z^A \frac{\partial f}{\partial z^A} = \ell f$$

for a function homogeneous of degree ℓ we may make the identifications

$$v^A \sim v^A + z^A \ . \tag{2.1}$$

In order to be tangent to the hypersurface $p^\alpha = 0$ the vector must satisfy

$$v^A \frac{\partial p^\alpha}{\partial z^A} = 0 \ , \tag{2.2}$$

which is compatible with (2.1) since $z^A \frac{\partial p^\alpha}{\partial z^A} = \deg(\alpha) p^\alpha = 0$ on $\mathcal{M}$. A deformation of this structure may be realised by replacing equation (2.2) by

$$v^A \left(\frac{\partial p^\alpha}{\partial z^A} + \mathsf{a}^\alpha_A \right) = 0 \tag{2.3}$$

where the a^α_A form a set of polynomials that are subject to the constraint

$$z^A \mathsf{a}^\alpha_A(z) = 0 \ .$$

It is then natural to set

$$a^\mu{}_\nu = -\frac{1}{2\pi i}\, \mathsf{a}^\alpha_\nu \chi^\mu{}_{\bar\rho\alpha} dx^{\bar\rho} \ , \tag{2.4}$$

where $\mathsf{a}^\alpha_\nu = \mathsf{a}^\alpha_A \frac{\partial z^A}{\partial x^\nu}$ is the projection of a^α_A along $\mathcal{M}$. This gives an explicit representation of the elements of $H^1_{\bar\partial}(\mathcal{M}, \mathrm{End}\, \mathcal{T})$ in terms of the polynomials a^α_A; a factor of $-1/2\pi i$ will simplify later expressions. The quantity $\chi^\mu{}_{\bar\rho\alpha}$ is the extrinsic curvature

$$\begin{aligned}
\chi_{\mu\nu}{}^\alpha &= \frac{\partial z^a}{\partial x^\mu} \frac{\partial z^b}{\partial x^\nu} \left(\frac{\partial^2 p^\alpha}{\partial z^a \partial z^b} - \Gamma^c_{ab} \frac{\partial p^\alpha}{\partial z^c} \right) \\
&= \frac{\partial z^A}{\partial x^\mu} \frac{\partial z^B}{\partial x^\nu} \frac{\partial^2 p^\alpha}{\partial z^A \partial z^B}
\end{aligned}$$

where the z^a are coordinates for the embedding space corresponding to taking $z^5 = 1$, say. In passing to the second equality we use the fact that in the Fubini-Study metric the term containing the connection does not contribute. The extrinsic curvature is perhaps more familiar from its occurrence in the representation

$$h^\mu = -\frac{1}{2\pi i}\, q^\alpha \chi_{\bar\nu}{}^\mu{}_\alpha dx^{\bar\nu} \ ,$$

where elements of $H^1_{\bar\partial}(\mathcal{M}, \mathcal{T})$ are represented in terms of the polynomials q^α [9,10].

2.2. The number of singlets

In the previous section we found a very convenient way of representing (some of) the **1**'s. However, as for the moduli fields associated to the **27**'s, one needs to consider the full Koszul complex and its corresponding spectral sequence in order to obtain the full group $H^1(\mathcal{M}, \mathrm{End}T)$. Though, unlike for the **27**'s and the $\overline{\mathbf{27}}$'s, the number of **1**'s is *not* constant over the space of complex structures (see Ref.[11,12])—even at the classical level. For certain special choices of the complex structure parameters, the number of the E_6 **1**'s which correspond to elements of $\mathrm{H}^1(\mathcal{M}, \mathrm{End}\,\mathcal{T})$ is larger than for a generic choice of these parameters.

As an example of this phenomenon let us consider the following Calabi–Yau manifold,

$$\mathcal{M} \in \begin{matrix} \mathbb{P}_3 \\ \mathbb{P}_2 \end{matrix}\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \quad : \quad \begin{cases} f(x) = \displaystyle\sum_{i=0}^{3} x_i^3 & = 0\ . \\ g(x,y) = \displaystyle\sum_{i=1}^{3} x_i y_i^3 + a x_0 y_1 y_2 y_3 & = 0\ , \end{cases} \tag{2.5}$$

For a generic choice of f, g, and in particular for the above defining equations with $a \neq 0$, one can show that $\dim H^1(\mathcal{M}, \mathrm{End}\,\mathcal{T}) = 88$. However, when $a = 0$ the number of elements of $\mathrm{H}^1(\mathcal{M}, \mathrm{End}\,\mathcal{T})$ jumps to 108. In the explicit computation this is because certain maps vanish in the long exact cohomology sequence in which $\mathrm{H}^1(\mathcal{M}, \mathrm{End}\,\mathcal{T})$ is an element. (For more details, see pp.240 in [14].)

Evidently, the number of massless **1**'s can jump as we vary the the complex structure. It also turns out that the number can jump as we vary the Kähler structure, though the extra singlets which arise in that case have nothing to do with the classical $H^1(\mathcal{M}, \mathrm{End}\,\mathcal{T})$. In many cases, one can understand this jump in the number of massless **1**'s as being due to symmetries. For instance, Landau-Ginzburg models possess a discrete R-symmetry in spacetime (corresponding to the quantum symmetry of the Landau-Ginzburg orbifold). In many such theories, the number of massless **1**'s jumps at the Landau-Ginzburg locus. The existence of the extra massless **1**'s (which arise in twisted sectors, and hence transform non-trivially under the "quantum" discrete R-symmetry) can be understood very simply [15]. Were they not present, the R-symmetry would suffer from a gravitational anomaly in spacetime.

It should be emphasized that the number of massless **1**'s does not always jump at the Landau-Ginzburg locus – $\mathbb{P}_4[5]$ is an example where the number does not jump – but in those cases, the R-symmetry is non-anomalous without the need for extra **1**'s. Whenever the R-symmetry would be anomalous without them, however, extra massless **1**'s appear at the Landau-Ginzburg locus to cancel the anomaly.

In the handful of explicitly worked examples, including the one discussed above, the jump in the number of massless **1**'s as one varies the complex structure also seems to

be associated with occurrence of discrete R-symmetries. In (2.5), for $a = 0$, there is a symmetry $R : x_0 \to \alpha x_0 , \alpha^3 = 1$. (This is an R-symmetry since the holomorphic 3-form transforms non-trivially.) It seems likely that cancelling the would-be-anomaly in the discrete R-symmetry is the "explanation" for why the extra massless **1**'s appear in all these cases [2]. It is not clear, however, that this one mechanism will account for all cases where the number of massless **1**'s jumps. Perhaps there are other, as yet undiscovered, mechanisms at work.

Finally, it should be clear that a physically complete moduli space for Calabi–Yau compactification[3] must be spanned by the moduli corresponding to the **27**'s and $\overline{\mathbf{27}}$'s, and all the massless **1**'s with exactly flat potential.

What are we to make of the extra massless **1**'s which arise on certain codimension-1 loci in the (2,2) moduli space? In most cases, one expects that they, though massless, do not (even classically) have a flat superpotential. In the language of the next subsection, they represent infinitesimal, but not integrable, deformations. In some cases, however, it is possible that the locus in question is a multicritical point, where two different branches (of, possibly, different dimensions) of the moduli space meet. In any case, we will restrict ourselves to one branch of the moduli space, and so focus on those **1**'s which are (classically) massless for all values of the complex structure.

2.3. Integrability and the vanishing of the $\mathbf{1}^3$ coupling

Elements of $s^\mu{}_\nu \in H^1(\mathcal{M}, \mathrm{End}T)$ correspond to first order deformations of the holomorphic structure of the tangent bundle to $\mathcal{M}$. Such a deformation can be thought of [8] as defining a deformed $\bar\partial$-operator,

$$(\bar D)^\mu{}_\nu = \delta^\mu{}_\nu \bar\partial + s^\mu{}_\nu(\epsilon)\wedge \tag{2.6}$$

acting on sections of T. $s^\mu{}_\nu(\epsilon)$ can be expanded in powers of ϵ,

$$s(\epsilon) = \epsilon^i s_i + \epsilon^i \epsilon^j s_{ij} + \dots \tag{2.7}$$

Demanding that $\bar D^2 = 0$, one finds, to first order in ϵ, that $(s_i)^\mu{}_\nu \in H^1(\mathcal{M}, \mathrm{End}\, T)$. To second order, one finds the condition,

$$\bar\partial s_{ij} + [s_i, s_j] = 0$$

[2] Note, again, that a discrete R-symmetry *by itself* does not imply the existence of extra massless **1**'s; the symmetry could be non-anomalous without additional massless fields. For example, in the $\mathbb{P}_4[5]$ family of models, the number of **1**'s is constant, regardless of R-symmetries.

[3] In general, it is only natural to consider the (2,2)-supersymmetric "standard" Calabi–Yau Ansatz merely as a special subset of the more general (0,2)-supersymmetric framework.

(where $[\cdot,\cdot]$ means commutator as elements of End T, and wedge-product as (0,1)-forms, and hence is symmetric in its arguments.) In order for the deformation to exist to second order, $[s_i, s_j]$ must be $\bar{\partial}$-exact. At each order in ϵ one finds a new potential obstruction, which must be trivial as an element of $H^2(\mathcal{M}, \mathrm{End}\, T)$ for the deformation to be integrable to that order.

If *all* of the obstructions vanish, then we say that the deformation is integrable, and, in that case, it is possible to define a covariant derivative on the space of parameters of the complex structure of the tangent bundle. It can then be shown [16] that the singlets can be represented as derivatives of a background gauge field,

$$s_i = \mathcal{D}_i a$$

and that

$$\bar{\partial}(\mathcal{D}_i s_j) = [s_i, s_j] \ . \tag{2.8}$$

Contracting the fermions, one readily sees that the classical contribution to the Yukawa coupling is

$$y(s_i, s_j, s_k) = \int_{\mathcal{M}} \Omega \wedge \mathrm{Tr}(s_i[s_j, s_k]) \tag{2.9}$$

From the above analysis, one sees that this vanishes precisely when the deformation is integrable to second order [3,10]. Presumably this correspondence persists to higher order, if the deformation is integrable to n^{th} oder, then the classical contribution to the spacetime superpotential vanishes through order $(n+1)$, and vice versa.

Singlets corresponding to polynomial deformations are, manifestly, integrable. Hence, it comes as no surprise that, using the tensor formalism developed in [12,14], one can show directly that if the three $\mathbf{1}$'s correspond to polynomial deformations then the $\mathbf{1}^3$ coupling vanishes. In particular, for $\mathbb{P}_4[5]$ all E_6 singlets are polynomial deformations and, as discussed previously, they may be thought of as the deformations of the 1st differential of the defining polynomial, which are not 1st differentials of the deformations of the defining polynomials [14]:

$$\vartheta(x) \sim \delta \mathrm{d} f(x) = \delta \mathrm{d}x^a f_{abcde} x^b x^c x^d x^e = \mathrm{d}x^a \vartheta_{a(bcde)} x^b x^c x^d x^e \ .$$

They are therefore represented by the tensor $\vartheta_{a(bcde)}$, which is symmetric in $bcde$, but vanishes upon symmetrization on all five indices. Next note that the $\mathbf{1}^3$ coupling requires one holomorphic 3-form, represented by one ϵ-tensor. The cubic coupling is therefore the product

$$\epsilon_{a_1 b_1 c_1 d_1 e_1}\ \vartheta^{(1)}_{a_2 b_2 c_2 d_2 e_2}\ \vartheta^{(2)}_{a_3 b_3 c_3 d_3 e_3}\ \vartheta^{(3)}_{a_4 b_4 c_4 d_4 e_4}\ ,$$

which is impossible to make into an invariant. This may be seen in a number of ways the simplest being that the five indices of $\epsilon_{a_1 b_1 c_1 d_1 e_1}$ must be contracted with indices of different $f^{ab\ldots c}$'s, at least five $f^{ab\ldots c}$'s must be used. This however provides 25 superscripts, to be contracted with 20 subscripts which is impossible, and so the coupling has to vanish.

Distler and Kachru [17] have shown that, for arbitrary choice of defining polynomial $W(x)$, the singlets correspond to 200 exactly flat directions in the spacetime superpotential, at the Landau-Ginzburg point. These flat directions break (2,2) worldsheet supersymmetry to (0,2), but it is not immediately clear whether these flat directions persist *away* from the Landau-Ginzburg point. Distler and Kachru gave an indirect argument that this *is* the case, and this claim is further explored by Silverstein and Witten [4,18] who argue that in fact the remaining 24 **1**'s are associated with flat directions as well. The present paper can be seen as an attempt to test the hypothesis from the opposite, Calabi–Yau (large radius) phase, by probing the instanton corrections to the singlet superpotential.

2.4. The vanishing of the $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ *coupling*

We have seen above that whenever the classical part of the $\mathbf{1}^3$ coupling vanishes, the deformations of $H^1_{\bar\partial}(\mathcal{M}, \mathrm{End}\mathcal{T})$ are integrable. It is tempting to speculate, based on this, that the classical part of the $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ coupling should also vanish. However in this case we do not have a deformation-theoretic interpretation of the coupling analogous to the previous case so we limit ourselves to a discussion of simple cases for which the $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ coupling can be shown to vanish, and the limitations of these arguments.

One of the elements of $H^1_{\bar\partial}(\mathcal{M}, \mathcal{T}^*)$ that is always present is the one corresponding to the Kähler form on $\mathcal{M}$ itself,

$$b_\nu = g_{\nu\bar\sigma} dx^{\bar\sigma} \ .$$

If in addition the **1** also corresponds to a polynomial deformation then we find

$$\int_{\mathcal{M}} \Omega \wedge h^\mu \wedge b_\nu \wedge s^\nu{}_\mu = -\int_{\mathcal{M}} \Omega \wedge h^\mu \mathsf{a}^\alpha_\mu \, \chi_{\bar\rho\bar\sigma} dx^{\bar\rho} \wedge dx^{\bar\sigma} = 0 \ ,$$

the last equality being due to the fact that $\chi_{\rho\sigma}{}^\alpha$ is symmetric in its lower indices. Thus the coupling vanishes for the case that the **1** corresponds to a polynomial deformation and the $\overline{\mathbf{27}}$ corresponds to the Kähler-form on $\mathcal{M}$. However, the Kähler-form on $\mathcal{M}$ may be written as $b_\nu = \sum_i v^i \omega_{\nu,i}$, where $\omega_{\nu,i}$ form a basis for $H^1_{\bar\partial}(\mathcal{M}, \mathcal{T}^*)$ and v^i are real parameters subject to some (finite number of) open conditions such as to form the Kähler cone. Therefore (with some abuse of notation), $\sum_i v^i \, (\mathbf{27}.\omega_{\nu,i}.\mathbf{1}) = 0$. On the other hand, the v^i are linearly independent, hence $(\mathbf{27}.\omega_{\nu,i}.\mathbf{1}) = 0$ for all i.

Thus, all classical $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ Yukawa couplings vanish whenever the **1**'s can be represented by polynomial deformations. Moreover, for the special choice of the $\overline{\mathbf{27}}$ which corresponds to the large radius limit (opposite to the Landau-Ginzburg point), these mixed Yukawa couplings vanish exactly.

3. **Preliminaries Concerning Instanton Contributions to Correlation Functions**

3.1. The zero modes

The sigma-model action at string tree level is given by

$$
\begin{aligned}
S \;=\; \int d^2z \Big\{ & g_{\mu\bar\nu}(\partial_z X^\mu \partial_{\bar z} X^{\bar\nu} + \partial_{\bar z} X^\mu \partial_z X^{\bar\nu}) + \psi_{\bar\mu}\left[\partial_z \psi^{\bar\mu} + \Gamma^{\bar\mu}_{\bar\nu\bar\rho}(\partial_z X^{\bar\nu})\psi^{\bar\rho}\right] \\
& + \lambda_\nu\left[\partial_{\bar z}\lambda^\nu + \Gamma^\nu_{\rho\sigma}(\partial_{\bar z}X^\rho)\lambda^\sigma\right] + R^\mu{}_\nu{}^{\bar\rho}{}_{\bar\sigma}\lambda_\mu\lambda^\nu\psi_{\bar\rho}\psi^{\bar\sigma}\Big\}\ .
\end{aligned}
\tag{3.1}
$$

The instantons correspond to mappings such that

$$
\partial_{\bar z} X^\mu \;=\; 0 \quad , \quad \partial_z X^{\bar\nu} \;=\; 0 \tag{3.2}
$$

and

$$
\begin{aligned}
\partial_{\bar z}\bar\psi^\mu &= 0\ , & \partial_{\bar z}\lambda^\mu &= 0\ ,\\
\partial_{\bar z}\bar\psi_\nu &= 0\ , & \partial_{\bar z}\lambda_\nu &= 0\ .
\end{aligned}
\tag{3.3}
$$

where we have conjugated the ψ equations to make the point that we are seeking elements of the Dolbeault groups $H^0(L, \mathcal{S}\otimes\mathcal{T}_L)$ and $H^0(L, \mathcal{S}\otimes\mathcal{T}_L^*)$, in which $\mathcal{S}$ denotes the spin bundle and $\mathcal{T}_L$ and $\mathcal{T}_L^*$ are the holomorphic tangent and cotangent bundles of L. Note that the fermionic equations do not contain terms involving the connection in virtue of the bosonic equations (3.2). The bosonic equations are the statement that the embedding of the worldsheet in the manifold is holomorphic. In the neighbourhood of the instanton we may choose local coordinates on the manifold such that $X^3 = \zeta$ is along the direction of the instanton, and ξ and η are sections of the normal bundle, as indicated in Figure 3.1.

Now it is standard analysis that the tangent bundle of the manifold decomposes into the tangent bundle and the normal bundle to the instanton

$$
\mathcal{T} \;=\; \mathcal{T}_L \oplus \mathcal{N}
$$

and owing to the fact that the instanton is a holomorphic submanifold this decomposition is holomorphic. Furthermore any holomorphic vector bundle over a sphere decomposes holomorphically into a direct sum of line bundles, $\mathcal{O}(k)$, that are classified by integers, k, corresponding to their first Chern class. For the tangent bundle to L we have

$$
\mathcal{T}_L \;=\; \mathcal{O}(2)
$$

since $c_1(\mathcal{T}_L) = \chi(S^2) = 2$. Decomposing also the normal bundle $\mathcal{N}_L$ we have

$$
\mathcal{T} \;=\; \mathcal{O}(p)\oplus\mathcal{O}(q)\oplus\mathcal{O}(2)\ .
$$

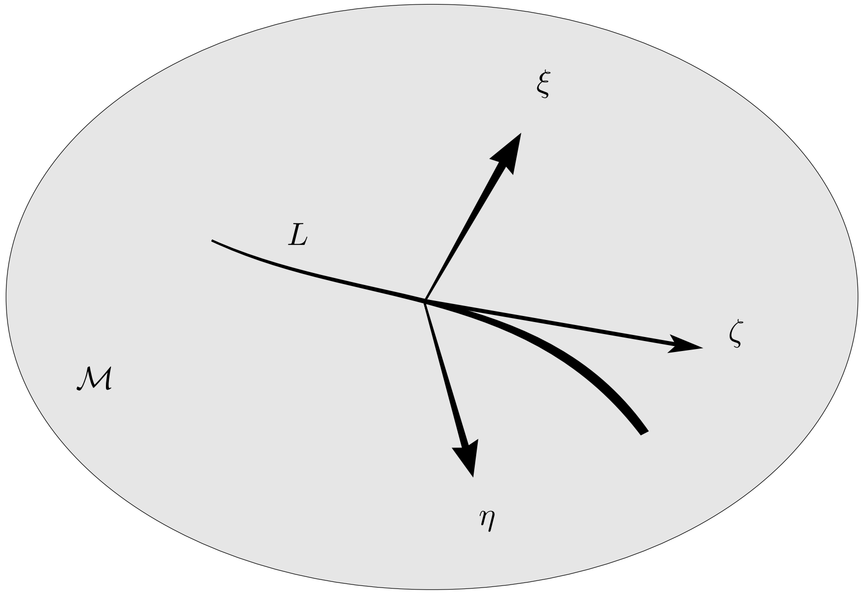

Figure 3.1: The coordinates on the Calabi–Yau manifold $\mathcal{M}$ in the neighbourhood of the instanton L.

Since also $c_1(\mathcal{T}) = 0$, in virtue of the fact that the manifold is Calabi–Yau, we have also that

$$p + q \ = \ -2 \ .$$

Various values of p and q are possible though for most manifolds $p = q = -1$ is the generic case, in the sense that for a generic choice of parameters all the lines are discrete (they do not lie in continuous families) and have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The issues of discreteness and splitting type are intimately related since a line which has splitting type other than $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ has deformations, *i.e.* it moves in a continuous family at least infinitesimally. We shall first take the normal bundle to be $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and for this case the map from the worldsheet to the instanton is of the form

$$X^3 = \zeta = \frac{az+b}{cz+d} \qquad , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \qquad . \tag{3.4}$$

Returning to equations (3.3) we now need to know only that $\mathcal{S} = \mathcal{O}(-1)$ and that the dimension of $H^0(S^2, \mathcal{O}(k))$ is given as a special case of the Bott–Borel–Weil theorem by

$$\mathrm{dim} H^0(S^2, \mathcal{O}(k)) \ = \ \begin{cases} k+1 \ , & \text{for } k \geq 0 \\ 0 \ , & \text{for } k \leq -1 \ . \end{cases}$$

For the λ^μ equation, for example, we are concerned with

$$\mathcal{S}\otimes\mathcal{T} \;=\; \mathcal{O}(-1)\otimes\Big(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}(2)\Big) \;=\; \mathcal{O}(-2)\oplus\mathcal{O}(-2)\oplus\mathcal{O}(1)$$

and we see that there are two zero modes for λ^ζ but none for λ^ξ or for λ^η. Proceeding in this way we find the only zero modes correspond to the components:

$$\begin{aligned}
\psi_{\bar\xi} &= \frac{\overline{\alpha}}{(\overline{cz}+\overline{d})}\ , &\qquad \lambda_\xi &= \frac{\alpha}{cz+d}\ ,\\
\psi_{\bar\eta} &= \frac{\overline{\beta}}{(\overline{cz}+\overline{d})}\ , &\qquad \lambda_\eta &= \frac{\beta}{cz+d}\ ,\\
\psi^{\bar\zeta} &= \frac{\overline{\gamma}}{(\overline{cz}+\overline{d})}+\frac{\overline{\delta}/\overline{c}}{(\overline{cz}+\overline{d})^2}\ , &\qquad \lambda^\zeta &= \frac{\gamma}{cz+d}+\frac{\delta/c}{(cz+d)^2}\ .
\end{aligned} \tag{3.5}$$

These results agree with those of [6,7].

3.2. Vertex operators and powers of g

The vertex operators for the **27**- and $\overline{\mathbf{27}}$-fields are conveniently written in terms of their $SO(10)\times U(1)\subset E_6$ content. First, we list the vertex operators for the available components of **27**, $\overline{\mathbf{27}}$ and the **1** of E_6, all in the ghost-number-(-1)-picture and at zero momentum :

$$V^I_{(-1)} \;=\; \mathrm{e}^{-\phi(\overline{z})}\, h_{\overline{\mu}}{}^{\alpha}(X)\,\psi^{\overline{\mu}}(\overline{z})\,\lambda^I(z)\,\lambda_\alpha(z)\ , \tag{3.6}$$

and

$$V^0_{(-1)} \;=\; \mathrm{e}^{-\phi(\overline{z})}\,\Omega_{\alpha\beta\gamma}\, h_{\overline{\mu}}{}^{\alpha}(X)\,\psi^{\overline{\mu}}(\overline{z})\,\lambda^\beta(z)\,\lambda^\gamma(z) \tag{3.7}$$

are the vertex operators for the (**10**,**1**) and the (**1**,**-2**) component, respectively, of the **27** vertex operator. The $\Omega_{\alpha\beta\gamma}$ are the components of Ω, the holomorphic three-form. Similarly,

$$V^{\bar J}_{(-1)} \;=\; \mathrm{e}^{-\phi(\overline{z})}\, b_{\alpha\overline{\mu}}(X)\,\psi^{\overline{\mu}}(\overline{z})\,\lambda^{\bar J}(z)\,\lambda^\alpha(z)\ , \tag{3.8}$$

and

$$V^{\overline{0}}_{(-1)} \;=\; \mathrm{e}^{-\phi(\overline{z})}\,\omega^{\alpha\beta\gamma}\, b_{\alpha\overline{\mu}}(X)\,\psi^{\overline{\mu}}(\overline{z})\,\lambda_\beta(z)\,\lambda_\gamma(z) \tag{3.9}$$

are the vertex operators for the (**10**,**-1**) and the (**1**,**2**) component, respectively, of the $\overline{\mathbf{27}}$ vertex operator. We define $\omega^{\alpha\beta\gamma}$ to be

$$\omega^{\alpha\beta\gamma} \stackrel{\text{def}}{=} \frac{g^{\alpha\bar\lambda}g^{\beta\bar\kappa}g^{\gamma\bar\sigma}\bar\Omega_{\bar\lambda\bar\kappa\bar\sigma}}{\|\Omega\|^2} \quad \text{where} \quad \|\Omega\|^2 \stackrel{\text{def}}{=} \frac{1}{3!}g^{\kappa\bar\lambda}g^{\mu\bar\nu}g^{\rho\bar\sigma}\Omega_{\kappa\mu\rho}\bar\Omega_{\bar\lambda\bar\nu\bar\sigma}\ . \tag{3.10}$$

The (**1**,**0**) vertex operator is

$$V^1_{(-1)} \;=\; \mathrm{e}^{-\phi(\overline{z})}\, s_{\overline{\mu}}{}^{\alpha}{}_{\beta}(X)\,\psi^{\overline{\mu}}(\overline{z})\,\lambda^\beta(z)\,\lambda_\alpha(z)\ . \tag{3.11}$$

Finally we record the ghost-number-(0)-picture dilaton vertex operator

$$V_R = ig_{\mu\bar{\nu}}(\partial x^{\mu})\Omega^{\bar{\nu}\bar{\rho}\bar{\sigma}}\psi_{\bar{\rho}}\psi_{\bar{\sigma}} \ . \tag{3.12}$$

which comes from the pull-back of the Kähler form to the world–sheet.

The nonrenormalization theorem [3] says that, since the spacetime superpotential must be a holomorphic function of the complex Kähler modulus, and perturbative corrections to the sigma model depend on the sigma model coupling constant (the scale of the metric g), but not on the θ-angle, there can be no perturbative sigma model corrections, either to the tree level, or to the instanton calculation of the spacetime superpotential.

Let us see how the counting of powers of the metric, g, goes, and verify that, indeed, the terms that appear in the spacetime superpotential appear at order g^0. We use the conventions in which the fermions have their indices raised and lowered: $\lambda^{\mu}, \lambda_{\nu}, \psi^{\bar{\mu}}, \psi_{\bar{\nu}}$. In these conventions, perturbation theory is not *manifestly* supersymmetric (as some of the indices on the fermions have been lowered), but this is more than compensated for by the fact that these conventions minimize the number of explicit factors of the metric which appear in the correlation functions. Since our interest will lie in calculating the spacetime superpotential, which receives no perturbative corrections anyway, little will be lost by not using a manifestly supersymmetric perturbation theory.

The forms $b_{\mu\bar{\nu}}(X)$, $h^{\mu}{}_{\bar{\nu}}(X)$, $s^{\mu}{}_{\nu\bar{\rho}}(X)$ which go into the definition of the vertex operators are taken to be g-independent. So, too, is $\Omega_{\mu\nu\rho}(X)$, the holomorphic three-form. The tensor $\omega^{\mu\nu\rho}$ defined in (3.10) above is also independent of g, as is clear from the relation

$$\Omega_{\mu\kappa\lambda}\omega^{\mu\nu\rho} = \delta^{\nu}_{\kappa}\delta^{\rho}_{\lambda} - \delta^{\rho}_{\kappa}\delta^{\nu}_{\lambda} \ .$$

However, $\Omega^{\bar{\mu}\bar{\nu}\bar{\rho}} \sim g^{-3}$, and $\omega_{\bar{\mu}\bar{\nu}\bar{\rho}} \sim g^3$. This scaling with g makes sense, as $\Omega_{\kappa\lambda\mu}\omega_{\bar{\nu}\bar{\rho}\bar{\sigma}}$ is the volume form on M, and so scales like g^3.

When the fields have zero modes, one gets powers of g in the path integral from doing the integration over the zero modes [19]. Each (complex) zero mode of X introduces a factor of g into the path integral. Each zero mode of λ^{μ} introduces a factor of $g^{-1/2}$ (and similarly for $\psi^{\bar{\mu}}$), but each zero mode of λ_{ν} introduces a factor of $g^{1/2}$ (and similarly for $\psi_{\bar{\nu}}$), so the index theorem ensures that the *net* effect of all the fermi zero modes is to introduce no powers of g. Each bose propagator brings down a power of g^{-1}, but the fermi propagators have *no* powers of g associated to them. Bringing down the four-fermi term, $R^{\mu}{}_{\nu}{}^{\bar{\rho}}{}_{\bar{\sigma}}\lambda_{\mu}\lambda^{\nu}\psi_{\bar{\rho}}\psi^{\bar{\sigma}}$ from the action introduces a factor of g^{-1}, as that is how $R^{\mu}{}_{\nu}{}^{\bar{\rho}}{}_{\bar{\sigma}}$ scales.

With these conventions, one easily checks that all of the 3-point functions we have ever considered (both at tree level, and at the n-instanton level) transform like g^0, as required. For instance, the $\overline{\mathbf{27}}^3$ coupling at tree level has an explicit g^{-3} from the vertex operators, since two of the vertex operators involve a $b_{\mu\bar{\nu}}$ and are g-independent, but one, for the auxiliary field in the $(\mathbf{1},\mathbf{2})$ of $SO(10)\times U(1)$, has a $b_{\mu\bar{\nu}}\omega^{\mu\kappa\lambda}\Omega^{\bar{\nu}\bar{\rho}\bar{\sigma}}$, and so scales[4]

[4] The spectral flow generator, which takes spacetime bosons into spacetime fermions (and spacetime fermions into the F-auxiliary field) scales like $g^{-3/2}$.

like g^{-3}. At tree level, there are 3 bose zero modes which leads to a factor of g^3 from the zero mode integrals, which yields $g^{-3+3}=1$. At the n-instanton level (an n-fold cover of a line [20]), there are $2n+1$ bose zero modes and we need to bring down $2n-2$ factors of the four-fermi interaction from the actions, yielding

$$g^{-3+(2n+1)-(2n-2)}=1$$

as required.

The counting for the singlet couplings is precisely analogous. The vertex operators in the correlation function scale as g^{-3}, as one of them is an F-auxiliary field (which carries a factor of g^{-3}), and the others are independent of g. No boson propagators are required, and the fermion dependence is such as to be able to absorb the zero modes present in an instanton background.

3.3. Decomposition of singlets along a line

Now a form $a^{\mu}{}_{\nu}=a_{\bar{\zeta}}{}^{\mu}{}_{\nu}d\bar{\zeta}\in H^1_{\bar{\partial}}(\mathcal{M},\mathrm{End}\,\mathcal{T})$ decomposes into components that transform in line bundle $\mathcal{O}(k)$ for $k=-3,0,3$,

$$\begin{array}{ll} a^{\zeta}{}_{\zeta} & \mathcal{O}(0) \\ a^{\zeta}{}_{j} & \mathcal{O}(3) \\ a^{i}{}_{\zeta} & \mathcal{O}(-3) \\ a^{i}{}_{j} & \mathcal{O}(0)\ . \end{array} \tag{3.13}$$

The decomposition of the tangent space to the manifold at a point of the instanton

$$\mathcal{T}=\mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}(2)$$

is the statement that the normal (upper) indices ξ and η count as "charge"-1, while the tangential index ζ has "charge"$+2$. Lower indices have the opposite charge. These cases are covered by another special case of the Bott–Borel–Weil theorem, which states that

$$\mathrm{dim}H^1_{\bar{\partial}}(S^2,\mathcal{O}(k))=\begin{cases}0, & \text{for } k\geq -1;\\ -k-1, & \text{for } k\leq -2.\end{cases}$$

Thus the components of $a^{\mu}{}_{\nu}$ are exact apart from the $a^{i}{}_{\zeta}$, so in Eq. (6.4) we may write

$$a_{\bar{\zeta}}{}^{i}{}_{j}d\bar{\zeta}=\bar{\partial}\alpha^{i}{}_{j}\quad\text{ and }\quad a_{\bar{\zeta}}{}^{\zeta}{}_{\zeta}d\bar{\zeta}=\bar{\partial}\alpha\ . \tag{3.14}$$

We shall be concerned with these parts of the singlets in Section 6 (the part $a^{\zeta}{}_{j}$ is also exact but plays no further rôle in our analysis). The components $a^{j}{}_{\zeta}$ represent a nontrivial cohomology group on the instanton. It is natural to associate with $a^{j}{}_{\zeta}$ a (1,1)-form

$$a^{j}\ \stackrel{\text{def}}{=}\ a_{\bar{\zeta}}{}^{j}{}_{\zeta}d\zeta d\bar{\zeta}$$

which for each value of j takes values in $\mathcal{O}(-1)$. Alternatively we may regard a^j as a (0,1)-form with values in $\mathcal{O}(-1) \otimes \mathcal{T}_L^* = \mathcal{O}(-3)$, this being just a repetition of the statement in table (3.13). According to the Bott–Borel–Weil Theorem, the cohomology group has dimension two. Stated differently: since a^j takes values in $\mathcal{O}(-1)$ there are two ways to integrate a^j over L to get a number. There are two sections (x^0, x^1) of $\mathcal{O}(1)$ over L which are the homogeneous coordinates of L thought of as $\mathbb{P}_1$. The product $a^j x^\alpha$ takes values in the trivial bundle and so may be integrated over L. Taking $\zeta^\alpha = (1, \zeta)$ as local sections of $\mathcal{O}(1)$, the burden of these remarks is that given a line L together with an a^j the essential information is encoded in a 2×2 matrix

$$A^{j\alpha} \stackrel{\text{def}}{=} \int_L a^j \zeta^\alpha \ . \tag{3.15}$$

The matrix $A^{j\alpha}$ has indices of two distinct types. We are free to redefine the coordinates (ξ, η)

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \longrightarrow \begin{pmatrix} \hat{\xi} \\ \hat{\eta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} .$$

The normal bundle is spanned by $d\xi \wedge d\eta$ and this is unchanged if the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has unit determinant. Similarly we may redefine the basis ζ^α by an independent $\mathrm{SL}(2, \mathbb{C})$ transformation. Under these transformations the matrix of a^j transforms according to the rule

$$A^{j\alpha} \longrightarrow \hat{A}^{j\alpha} = U^j{}_k A^{k\beta} V_\beta{}^\alpha$$

with independent $\mathrm{SL}(2, \mathbb{C})$ matrices acting on the left and on the right. Invariants may be formed by contracting indices with the permutation symbols ε_{jk} and $\varepsilon_{\alpha\beta}$. The invariant product of two matrices is

$$\varepsilon_{jk} \varepsilon_{\alpha\beta} A^{j\alpha} B^{k\beta} \ .$$

From these simple observations, we conclude that each isolated line can contribute to at most four eigenvalues of the mass matrix. Also, much of the analysis of this article relevant to the Yukawa couplings follows from the simple fact that there is no invariant combination of three matrices.

4. **Masses Generated by Instantons**

4.1. The mass matrix

We wish to evaluate the correlation function corresponding to a dilaton–singlet–singlet coupling $m(a,b) = \langle V_R V_a V_b \rangle$ where V_R corresponds to a dilaton and V_a and V_b are vertex operators corresponding to the singlets associated to forms $a^\mu{}_\nu$ and $b^\mu{}_\nu$. By referring back to Section 3.2 we see that we are to evaluate the expression:

$$\begin{aligned} \langle V_R V_a V_b \rangle = \mu \int d^6 z d^4 \alpha d^4 \bar{\alpha} & \Big[i g_{\mu\bar{\nu}} (\partial x^\mu) \Omega^{\bar{\nu}\bar{\rho}\bar{\sigma}} \psi_{\bar{\rho}} \psi_{\bar{\sigma}} \Big]_{(1)} \\ & \times \Big[\mathrm{e}^{-\phi}\, a_{\bar{\kappa}}{}^{\alpha}{}_{\beta}\, \psi^{\bar{\kappa}}\, \lambda^{\beta}\, \lambda_{\alpha} \Big]_{(2)} \\ & \times \Big[\mathrm{e}^{-\phi}\, b_{\bar{\lambda}}{}^{\gamma}{}_{\delta}\, \psi^{\bar{\lambda}}\, \lambda^{\delta}\, \lambda_{\gamma} \Big]_{(3)} \ , \end{aligned} \tag{4.1}$$

where we have written V_R, the vertex operator for the F–auxiliary field, in the 0–picture. The factor μ, defined as

$$\mu \stackrel{\text{def}}{=} \left(\frac{\det(G^B)}{\mathrm{Det}'(\bar{\partial}_{\mathcal{T}}^\dagger \bar{\partial}_{\mathcal{T}})} \right) \left(\frac{\mathrm{Det}'(\bar{\partial}_{\mathcal{T}\otimes\mathcal{S}}^\dagger \bar{\partial}_{\mathcal{T}\otimes\mathcal{S}})}{\det(G^F)} \right) \ , \tag{4.2}$$

represents the contributions from the integration over the non–zero modes in the path integral and also the measure of the parameter space of the zero modes. The first factor is the contribution of bosonic modes and the second factor is the contribution of fermionic modes. Here, G^B is the metric on the parameter space of bosonic zero modes and G^F the metric on the parameter space of fermionic zero modes.

We will now give a simple argument why μ is constant in the basis for the zero modes (3.5). (For a more detailed discussion of the values of the individual factors in (4.2) see Appendix A.) This follows from the Distler–Greene [7] computation of the instanton contribution to the $\overline{\mathbf{27}}^3$ Yukawa coupling since determinantal factors are the same in each case (Distler and Greene use the same basis of fermionic zero modes that we have adopted here). The vertex operators for the $\overline{\mathbf{27}}^3$ coupling do not introduce any dependence on complex structure parameters and the final expression for the Yukawa coupling calculated by Distler and Greene does not depend on the complex structure parameters. Indeed, if one calculates the ratio of the singlet mass and the $\overline{\mathbf{27}}^3$ coupling, the determinants in question appear both in the numerator and denominator and cancel out. Similarly, the right-moving parts of the correlation function are identical. Indeed, the only difference between the two calculations occurs in the left-moving part of the correlation function.

The evaluation of the zero mode integration in expression (4.1) is elementary given the table (3.5) which records the nonzero components of the zero modes. Thus the first expression in square brackets in (4.1) simplifies to

$$\int d\overline{\beta}d\overline{\alpha}\ ig_{\zeta\bar{\zeta}}\frac{1}{(cz_1+d)^2}\varepsilon^{\bar{\imath}\bar{\jmath}}\psi_{\bar{\imath}}(\overline{z}_1)\psi_{\bar{\jmath}}(\overline{z}_1)=2ig_{\zeta\bar{\zeta}}\frac{1}{|cz_1+d|^4}\ .$$

From the second and third expressions we have a factor

$$\begin{aligned}\int d\overline{\delta}\,d\overline{\gamma}\ \psi^{\bar{3}}(\overline{z}_2)\psi^{\bar{3}}(\overline{z}_3)&=\int d\overline{\delta}\,d\overline{\gamma}\left(\frac{\overline{\gamma}}{(\overline{c}\overline{z}_2{+}\overline{d})}+\frac{\overline{\delta}/\overline{c}}{(\overline{c}\overline{z}_2{+}\overline{d})^2}\right)\left(\frac{\overline{\gamma}}{(\overline{c}\overline{z}_3{+}\overline{d})}+\frac{\overline{\delta}/\overline{c}}{(\overline{c}\overline{z}_3{+}\overline{d})^2}\right)\\&=\frac{(\overline{z}_2-\overline{z}_3)}{(\overline{c}\overline{z}_2+\overline{d})^2\,(\overline{c}\overline{z}_3+\overline{d})^2}\ ,\end{aligned}\tag{4.3}$$

which cancels the factor $(\overline{z}_2-\overline{z}_3)^{-1}$ from the free correlator $\left\langle\, \mathrm{e}^{-\phi(\overline{z}_2)}\,\mathrm{e}^{-\phi(\overline{z}_3)}\,\right\rangle$.

Proceeding in this manner we obtain an integral which is most simply written in terms of the variable $\zeta=\frac{az+b}{cz+d}$:

$$\begin{aligned}m(a,b)\overset{\text{def}}{=}\langle V_R V_a V_b\rangle&=\mu\int_L J\ \times\ \int_{L\times L}\varepsilon_{jk}a^j(\zeta_2)b^k(\zeta_3)(\zeta_2-\zeta_3)\\&=\mu\,\varepsilon_{jk}\varepsilon_{\alpha\beta}A^{j\alpha}B^{k\beta}\int_L J\end{aligned}\tag{4.4}$$

where $J=ig_{\zeta\bar{\zeta}}d\zeta d\bar{\zeta}$ is the Kähler form restricted to L.

Given a basis a_I for $H^1(\mathcal{M},\mathrm{End}\,\mathcal{T})$ and an instanton L let us denote by the reduced mass matrix for L the matrix

$$m_{IJ}^{(L)}\ =\ m^{(L)}(a_I,a_J)\ =\ \varepsilon_{jk}\varepsilon_{\alpha\beta}\,A_I^{(L)\,j\alpha}A_J^{(L)\,k\beta}\ ,$$

with $A_I^{(L)\,j\alpha}$ the matrix of a_I along L. Note that, as expected from the discussion at the end of last Section, L can give mass to at most four singlets since among the matrices $A_I^{(L)}$ there can be at most four that are linearly independent.

4.2. Other splitting types

The above analysis refers to instantons with normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. For some manifolds this is the generic situation. For a generic quintic hypersurface in $\mathbb{P}_4[5]$ there are 2875 lines all of which have normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. Other cases are possible; the manifold may have instantons that form continuous families and for these instantons the normal bundle will be $\mathcal{O}(p)\oplus\mathcal{O}(-p-2)$ for some $p\geq 0$. It is known also, for example, that normal bundles of type $\mathcal{O}(0)\oplus\mathcal{O}(-2)$ and $\mathcal{O}(1)\oplus\mathcal{O}(-3)$ can arise in $\mathbb{P}_4[5]$ for certain choices of defining polynomial. It is instructive to see how the counting goes for these other cases.

$\underline{\mathcal{O}(0) \oplus \mathcal{O}(-2)}$

For this case we take the normal coordinate ξ to be a local section of $\mathcal{O}(0)$ and η to be a local section of $\mathcal{O}(-2)$. By considering the transformation properties of the forms $a^\mu = a^\mu{}_\zeta d\zeta$ as in Table (3.13) we see that a^ξ and a^η are nontrivial. The form a^ξ transforms in $\mathcal{O}(0)$ and has one degree of freedom while a^η transforms in $\mathcal{O}(-2)$ and has three degrees of freedom. In this case we again find a total of four degrees of freedom. These correspond to a scalar

$$A \;=\; \int_L a^\xi$$

and a symmetric matrix

$$A^{\alpha\beta} \;=\; \int_L a^\eta \zeta^\alpha \zeta^\beta \;, \qquad \alpha, \beta = 0, 1 \;.$$

Invariant bilinear products of two singlets on L are for example

$$AB \quad \text{and} \quad \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\, A^{\alpha\beta} B^{\gamma\delta} \;.$$

Note however that the coordinate ξ is not uniquely determined by the statement that it is a section of $\mathcal{O}(0)$; since if ξ is such a section then so is

$$\hat{\xi} \;=\; \xi + c_{\alpha\beta}\zeta^\alpha\zeta^\beta\eta$$

for any coefficients $c_{\alpha\beta}$. The change $\xi \to \hat{\xi}$ leads to a change in A

$$A \;\to\; A + c_{\alpha\beta}A^{\alpha\beta} \;.$$

Since the contribution of a line to the mass matrix cannot depend on our choice of coordinate ξ at first sight it would seem to be the case that the contribution to the mass matrix only involves the invariant quantity $\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}A^{\alpha\beta}B^{\gamma\delta}$. In fact, from the fermionic zero-mode computation in section 4.1 one may naively conclude that the mass matrix is zero for an $\mathcal{O}(0) \oplus \mathcal{O}(-2)$ line. As will be discussed in the next section and the appendix the contribution is however non-zero. This is seen by studying the degeneration, parametrized by a parameter ϵ, of lines of type $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ which coincide in the limit $\epsilon \to 0$ to become an $\mathcal{O}(0) \oplus \mathcal{O}(-2)$ line. The understanding of these contributions is complicated by the existence of poles. We will find that each individual line contributes a term to the mass matrix which develops a pole as $\epsilon \to 0$ which is indeed of the form $\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}A^{\alpha\beta}B^{\gamma\delta}$. These poles however cancel when summed over the lines. There remains a finite contribution but this now involves further invariants formed from derivatives of the $A^{\alpha\beta}$.

$\underline{\mathcal{O}(p) \oplus \mathcal{O}(-p-2) \text{ for } p \geq 1}$

For these cases we take the coordinates ξ and η to be local sections of $\mathcal{O}(p)$ and $\mathcal{O}(-p-2)$. The form a^{ξ} is now trivial and we are thus left with a^{η} which transforms in $\mathcal{O}(-p-2)$ and has $p+3$ degrees of freedom corresponding to a tensor

$$A^{\alpha_1\alpha_2\ldots\alpha_{p+2}} = \int_L a^{\eta}\zeta^{\alpha_1}\zeta^{\alpha_2}\cdots\zeta^{\alpha_{p+2}} , \qquad \alpha_i = 0, 1 .$$

Note however that the invariant product between two of these

$$\varepsilon_{\alpha_1\beta_1}\varepsilon_{\alpha_2\beta_2}\ldots\varepsilon_{\alpha_{p+2}\beta_{p+2}}\ A^{\alpha_1\alpha_2\ldots\alpha_{p+2}}B^{\beta_1\beta_2\ldots\beta_{p+2}}$$

is symmetric for p even but antisymmetric for p odd so for instantons corresponding to odd p these terms cannot contribute to the mass matrix.

5. **Examples of the Effects of Lines in $\mathbb{P}_4[5]$**

5.1. General facts

Given a line, L, we may choose coordinates adapted to the line such that L corresponds to the equations $X_2 = X_3 = X_4 = 0$. In terms of these coordinates the quintic, p, that defines the Calabi–Yau manifold takes the form

$$p = X_2 F + X_3 G + X_4 H + K \ ,$$

F, G and H being quartics in X_1 and X_5, and K being a quintic of order 2 or more in X_2, X_3 and X_4. The splitting type of the normal bundle of L depends on the degree of independence of the quartics F, G and H. Katz [21] has shown that if there are no linear polynomials $\ell_1(X)$, $\ell_2(X)$, $\ell_3(X)$ such that

$$\ell_1(X)F + \ell_2(X)G + \ell_3(X)H = 0 \ ,$$

such a relation being termed a quintic relation, then the normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. If there is a quintic relation but no quartic relation

$$c_1 F + c_2 G + c_3 H = 0 \ ,$$

with constants c_1, c_2, c_3, then the normal bundle is of type $\mathcal{O}(0) \oplus \mathcal{O}(-2)$. The final case is if there is a quartic relation (*i.e.* F, G and H are not linearly independent) then the normal bundle is of type $\mathcal{O}(1) \oplus \mathcal{O}(-3)$. For $\mathbb{P}_4[5]$ no other case is possible [22] in virtue of the fact that the normal bundle of the line in $\mathbb{P}_4[5]$ is a subbundle of L when thought of as a line in $\mathbb{P}_4$. Since the normal bundle of the line in $\mathbb{P}_4$ is $\mathcal{O}(1)^3$, the normal bundle in $\mathbb{P}_4[5]$, $\mathcal{O}(e_1) \oplus \mathcal{O}(e_2)$, must have both $e_1, e_2 \leq 1$.

Examples of such functions are

$$F = X_1^2 X_5^2 \quad , \quad G = X_5^4 \quad , \quad H = X_1^4$$

for which there is no quintic relation and

$$F = X_1^3 X_5 \quad , \quad G = X_5^4 \quad , \quad H = X_1^4$$

for which there is a quintic relation but no quartic relation. We shall be concerned, in the following, with a degenerating family for which

$$F = X_1^3 X_5 + \epsilon X_1^2 X_5^2 \quad , \quad G = X_5^4 \quad , \quad H = X_1^4$$

such a line has normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ for each nonzero ϵ but the normal bundle is $\mathcal{O}(0)\oplus\mathcal{O}(-2)$ when $\epsilon=0$.

In order to find the normal coordinates ξ, η, we can follow the prescription of Harris [22]. We first look for sets of functions of weight one in X_1 and X_5, $\mathbf{m}=(m_2,m_3,m_4)$, such that

$$m_2F+m_3G+m_4H=0\ . \tag{5.1}$$

Each $\mathbf{m}$ will be used to build a normal coordinate. For a coordinate in an $\mathcal{O}(-n)$ bundle, $\mathbf{m}$ should be defined in the patch U_1 (*i.e.* when $X_1\neq 0$), and $\left(\frac{X_1}{X_5}\right)^n\mathbf{m}$ must be defined in U_5. These are then used to define the normal vectors in the appropriate patches. Since we have two normal coordinates, there are two sets of functions, $\mathbf{m}$ and $\mathbf{m}'$. Let (ξ,η) be the normal coordinates over U_1 and $(\tilde{\xi},\tilde{\eta})$ the normal coordinates over U_5. On U_1 we set

$$\begin{aligned}\frac{\partial}{\partial\xi}&=m_2\frac{\partial}{\partial X_2}+m_3\frac{\partial}{\partial X_3}+m_4\frac{\partial}{\partial X_4}\\ \frac{\partial}{\partial\eta}&=m'_2\frac{\partial}{\partial X_2}+m'_3\frac{\partial}{\partial X_3}+m'_4\frac{\partial}{\partial X_4}\end{aligned} \tag{5.2}$$

and on the intersection $U_1\cap U_5$ we find $\frac{\partial}{\partial\tilde{\xi}}=\left(\frac{X_1}{X_5}\right)^n\frac{\partial}{\partial\xi}$ and $\frac{\partial}{\partial\tilde{\eta}}=\left(\frac{X_1}{X_5}\right)^{n'}\frac{\partial}{\partial\eta}$ for an $\mathcal{O}(-n)\oplus\mathcal{O}(-n')$ bundle.

It is perhaps worth recording here also that since the only splitting types for $\mathbb{P}_4[5]$ are $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, $\mathcal{O}(0)\oplus\mathcal{O}(-2)$ and $\mathcal{O}(1)\oplus\mathcal{O}(-3)$ the data describing the restriction of the singlet to a line consists in each case of four numbers. These are the quantities $A^{j\alpha}$, A and $A^{\alpha\beta}$, and $A^{\alpha\beta\gamma}$ for the three cases respectively. We can understand this by noting that a singlet $\mathsf{a}_M dx^M$ when restricted to the line $X_2=X_3=X_4=0$ takes the form

$$\mathsf{a}_M dX^M=(\mathsf{A}_0X_1^3+\mathsf{A}_1X_1^2X_5+\mathsf{A}_2X_1X_5^2+\mathsf{A}_3X_5^3)(X_1dX_5-X_5dX_1) \tag{5.3}$$

which is specified by giving the four coefficients $\mathsf{A}_0,\ldots,\mathsf{A}_3$.

A remarkable fact demonstrated by Harris [22] is that an arbitrary permutation of the lines may be achieved by monodromy. That is, if the manifold is deformed around a closed loop in the moduli space then the manifold returns to the original manifold but with the lines permuted and an arbitrary permutation can be achieved by suitable choice of loop. Harris demonstrates this by showing that there are monodromies that interchange any two given lines while leaving the remaining lines unchanged.

For the case of $\mathbb{P}_4[5]$, we may choose a basis for the singlets that is independent of the complex structure. We may do this by forming from the set of all vectors $\mathsf{s}_A(X)$ of quartic monomials (of which there are $5\times 70=350$) fixed linear combinations that satisfy $X^A\mathsf{s}_A(X)=0$. We are left with a basis of 224 vectors of quartics with fixed coefficients. In this section we will first study the mass matrix corresponding to an $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ line. We will then study how the matrix behaves as three $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ lines come together to form an $\mathcal{O}(0)\oplus\mathcal{O}(-2)$ line. To this end we examine first the geometry of the normal bundle of an $\mathcal{O}(0)\oplus\mathcal{O}(-2)$ line and then we study the degeneration.

5.2. An $\mathcal{O}(0) \oplus \mathcal{O}(-2)$ line as the limit of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ lines

Consider the pair of lines in $\mathbb{P}_4$ given parametrically by

$$l_{\pm\epsilon} = (u,\ \pm\epsilon u,\ 0,\ \mp\epsilon v,\ v)\ , \tag{5.4}$$

where (u, v) are the homogeneous coordinates of the $\mathbb{P}_1$. These are described by the equations

$$\begin{aligned} \epsilon^2 X_1^2 - X_2^2 &= 0 \\ \epsilon^2 X_5^2 - X_4^2 &= 0 \\ X_1 X_4 + X_2 X_5 &= 0 \\ \epsilon^2 X_1 X_5 + X_2 X_4 &= 0 \\ X_3 &= 0\ , \end{aligned}$$

and so are embedded in any hypersurface $p = 0$ in $\mathbb{P}_4[5]$ with polynomial

$$p = (\epsilon^2 X_1^2 - X_2^2)Q + (\epsilon^2 X_5^2 - X_4^2)\tilde{Q} + (X_1 X_4 + X_2 X_5)S + (\epsilon^2 X_1 X_5 + X_2 X_4)R + X_3 G$$

with Q, $\tilde{Q}$, S and R cubics, G a quartic.

Firstly note that when $\epsilon = 0$, the lines coincide, and are given by $X_2 = X_3 = X_4 = 0$. Linearizing in X_2, X_3 and X_4, the polynomial p becomes

$$p = X_2 X_5 S + X_3 G + X_4 X_1 S\ ,$$

and so this line has normal bundle $\mathcal{O}(0) \oplus \mathcal{O}(-2)$ for sufficiently general S and G. We will choose

$$S|_{l_{\pm\epsilon}} = u^3\ ,\ G|_{l_{\pm\epsilon}} = v^4\ .$$

If p is sufficiently general then for $\epsilon \neq 0$ the lines have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Since the line l_ϵ is defined as $Y_2 = Y_3 = Y_4 = 0$ for the variables

$$\begin{aligned} Y_2 &= X_2 - \epsilon X_1 \\ Y_3 &= X_3 \\ Y_4 &= X_4 + \epsilon X_5 \end{aligned}$$

we linearize the polynomial in the Y's to find

$$p|_{l_\epsilon} = Y_2 F + Y_3 G + Y_4 H$$

with the quartic functions

$$F = X_5 S - 2\epsilon X_1 Q - \epsilon X_5 R \quad , \quad H = X_1 S + 2\epsilon X_5 \tilde{Q} + \epsilon X_1 R\ .$$

For definiteness, we choose a polynomial with

$$
\begin{aligned}
&Q = -\frac{1}{2}X_1X_5^2 - \frac{1}{2}\kappa X_5^3 + \Delta Q \quad ,\\
&\tilde{Q} = -X_1 T \quad ,\\
&S = X_1^3 - X_4 T - \frac{1}{4}(\epsilon^2 X_1 X_5 + X_2 X_4) X_5 + \Delta S \quad ,\\
&R = X_5 T \quad ,\\
&G = X_5^4 + \Delta G \ ,
\end{aligned}
$$

where T is a quadratic polynomial and ΔQ, ΔS and ΔG are generic polynomials of the appropriate degree which are are chosen to vanish on both lines $l_{\pm\epsilon}$. ΔQ, ΔS and ΔG are introduced only to ensure the transversality of p and are otherwise irrelevant in the discussion below.

The linearizations of p about the lines $l_{\pm\epsilon}$ are particularly simple. The linearization about l_ϵ takes the form

$$
p|_{l_\epsilon} = Y_2\,(X_1^3 X_5 + \epsilon X_1^2 X_5^2 + \kappa\epsilon X_1 X_5^3) + Y_3\, X_5^4 + Y_4\, X_1^4 \ , \tag{5.5}
$$

in which we see clearly that the normal bundle is $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ for $\epsilon \neq 0$ and is $\mathcal{O}(0)\oplus\mathcal{O}(-2)$ when $\epsilon = 0$.

The calculation of the contribution of each line to the mass matrix $\epsilon_{ij}\epsilon_{\alpha\beta}A^{i\alpha}B^{j\beta}$ is straightforward. We need to calculate the matrices $A^{i\alpha}$ given by

$$
A^{i\alpha} = \int_L a^j \zeta^\alpha \ ,
$$

where we use the representation for the singlets [9] as discussed in section 2:

$$
a^\mu{}_\nu = -\frac{1}{2\pi i}\,\mathsf{a}^\alpha{}_\nu\;\chi^\mu{}_{\bar\rho\alpha}\,\mathrm{d}X^{\bar\rho} = -\frac{1}{2\pi i}\,\mathsf{a}^\alpha{}_\nu\, h_{\alpha\bar\beta}\,\chi_{\bar\tau\bar\rho}{}^{\bar\beta}\,g^{\mu\bar\tau}\,\mathrm{d}X^{\bar\rho} \ . \tag{5.6}
$$

We need to find first the normal coordinates $(\xi,\,\eta)$, which as outlined in the previous subsection, is done by solving for $\mathbf{m} = (m_2, m_3, m_4)$ in $m_2F + m_3G + m_4H = 0$. Since the lines $l_{\pm\epsilon}$ are $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ for $\epsilon \neq\ 0$, the m's are all quadratic in ζ. We find then that in the coordinate patch with $X_1 \neq 0$ we have the following relations:

$$
l_{+\epsilon} \quad \left\{
\begin{aligned}
\frac{Y_2}{X_1} &= (-1+\kappa\zeta)\xi + \frac{1}{\epsilon(\epsilon-\kappa)}(1-\epsilon\zeta)\zeta\,\eta\\
\frac{Y_3}{X_1} &= -\kappa^2\epsilon\,\xi + \left(1 + \frac{\kappa\epsilon}{\epsilon-\kappa}\zeta\right)\eta\\
\frac{Y_4}{X_1} &= (1+(\epsilon-\kappa)\zeta)\zeta\,\xi - \frac{1}{\epsilon(\epsilon-\kappa)}\zeta^2\,\eta\\
\frac{X_5}{X_1} &= \zeta
\end{aligned}
\right. \tag{5.7}
$$

We work with the Fubini–Study metric on $\mathbb{P}_4$ which descends to a metric in the neighbourhood of the instanton as

$$g_{i\bar{\jmath}}\mathrm{d}X^i\mathrm{d}X^{\bar{\jmath}} = \left(\frac{\delta_{i\bar{\jmath}}}{\sigma} - \frac{X_i X_{\bar{\jmath}}}{\sigma^2}\right)\mathrm{d}X^i\mathrm{d}X^{\bar{\jmath}} \ , \tag{5.8}$$

where i and $\bar{\jmath}$ run over the values for the inhomogeneous coordinates, and $\sigma = \sum_A |X^A|^2$. The components of interest of the extrinsic curvature are calculated from [9]

$$\chi_{\mu\nu}{}^{\alpha} = \frac{\partial X^A}{\partial x^\mu}\frac{\partial X^B}{\partial x^\nu}\frac{\partial^2 p^\alpha}{\partial X^A \partial X^B} \ ,$$

as discussed in Section 2.1. For a polynomial p, the factor $h_{\alpha\bar{\beta}}$ (the index α takes only one value in our case and we now drop this index for notational simplicity) is given by

$$h^{-1} = \sigma \sum_A |\partial p/\partial X^A|^2 \ .$$

For the singlets that are nonzero along the instanton, the factor a_ν has the form $\mathsf{a}_\zeta = X_1^4\,\mathsf{A}(\zeta)$, with $\mathsf{A}(\zeta) = \sum_{i=0}^{3}\mathsf{A}_i\zeta^i$, which finds its origin in the assignment (5.3).

For the line l_ϵ the only relevant components of the extrinsic curvature are

$$\begin{aligned}
\chi_{\xi\zeta} &= X_1^5[-1 + (-2\epsilon + \kappa)\zeta - \kappa\epsilon\zeta^2 - \kappa^2\epsilon\zeta^3] \\
\chi_{\eta\zeta} &= X_1^5\frac{\zeta}{\epsilon(\epsilon - \kappa)}[1 + \epsilon\zeta + (2\epsilon - \kappa)\epsilon\zeta^2 + \kappa\epsilon^2\zeta^3] \ .
\end{aligned}$$

Putting all the information together we obtain the matrix for the singlets

$$A^{j\alpha}(l_\epsilon) = \begin{pmatrix} \dfrac{\mathsf{A}_2}{\epsilon(\epsilon-\kappa)} - \dfrac{\mathsf{A}_1}{(\epsilon-\kappa)} & \dfrac{\mathsf{A}_1}{\epsilon(\epsilon-\kappa)} - \dfrac{\mathsf{A}_0}{(\epsilon-\kappa)} \\ \mathsf{A}_3 - \kappa\mathsf{A}_2 & \mathsf{A}_2 - \kappa\mathsf{A}_1 \end{pmatrix} \ .$$

The surprising conclusion is that l_ϵ contributes a term to the mass matrix that has a pole as the lines coalesce at $\epsilon = 0$

$$m \ \sim \ \frac{1}{\epsilon(\epsilon-\kappa)}(\mathsf{A}_2^2 - \mathsf{A}_1\mathsf{A}_3) \ . \tag{5.9}$$

However, when the contribution of the pair of lines $l_{+\epsilon}$ and $l_{-\epsilon}$ is added up, the pole at $\epsilon = 0$ cancels except when $\kappa = 0$. In the case of $\kappa = 0$, the effect as $\epsilon \to 0$ is that the leading term of order $1/\epsilon^2$ is multiplied by two and the terms of $\mathcal{O}(\epsilon^{-1})$ cancel.

What is happening is that the polynomial p has a third line that when $\kappa = 0$ also coalesces with the lines $l_{\pm\epsilon}$ as $\epsilon \to 0$. For a certain choice of the polynomials ΔQ, ΔS and ΔG, this line is given by

$$l_{\epsilon^2} = \left(u,\ \frac{1}{2}\epsilon^2 v,\ 0,\ 0,\ v\right) \ . \tag{5.10}$$

The situation in this case is very interesting since now there are three lines that coalesce at $\epsilon = 0$. The line l_{ϵ^2} is defined as $Z_2 = Z_3 = Z_4 = 0$ for the variables

$$
\begin{aligned}
Z_2 &= X_2 - \frac{1}{2}\epsilon^2 X_5 \\
Z_3 &= X_3 \\
Z_4 &= X_4 \quad .
\end{aligned}
$$

Linearizing the polynomial in the Z's we find

$$
p|_{l_{\epsilon^2}} = Z_2 \left(X_1^3 X_5 + \frac{1}{4}\epsilon^2 X_1 X_5^3 \right) + Z_3\, X_5^4 + Z_4 \left(X_1^4 - \frac{1}{4}\epsilon^2 X_2 X_5^2 - \frac{1}{16}\epsilon^4 X_5^4 \right) \quad . \tag{5.11}
$$

from which we see that l_{ϵ^2} is indeed of type $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Following again the procedure above to find the contribution to the mass matrix for this line, we find for the normal coordinates (ξ, η)

$$
l_{\epsilon^2} \quad \left\{ \begin{aligned}
\frac{Z_2}{X_1} &= \left(-1 + \frac{\epsilon^2}{2}\zeta^2 \right) \xi - \frac{2}{\epsilon^2}\zeta\, \eta \\
\frac{Z_3}{X_1} &= -\frac{\epsilon^4}{16}\zeta\, \xi + \left(1 + \frac{\epsilon^2}{8}\zeta^2 \right) \eta \\
\frac{Z_4}{X_1} &= \zeta\, \xi + \frac{2}{\epsilon^2}\zeta^2\, \eta \\
\frac{X_5}{X_1} &= \zeta \ .
\end{aligned} \right. \tag{5.12}
$$

The only nonzero components of the extrinsic curvature are now

$$
\begin{aligned}
\chi_{\xi\zeta} &= -X_1^5 \frac{1}{8} \left(8 + 6\epsilon^2\zeta^2 + \epsilon^4\zeta^4 \right) \\
\chi_{\eta\zeta} &= -X_1^5 \frac{\zeta}{2\epsilon^2} \left(4 - 3\epsilon^2\zeta^2 \right) \quad .
\end{aligned}
$$

The coefficient matrix for the singlets is now

$$
A^{j\alpha}(l_{\epsilon^2}) = \begin{pmatrix} -2\frac{\mathsf{A}_2}{\epsilon^2} - \frac{\mathsf{A}_0}{2} & -2\frac{\mathsf{A}_1}{\epsilon^2} \\ \mathsf{A}_3 - \frac{\epsilon^2 \mathsf{A}_1}{4} & \mathsf{A}_2 - \frac{\epsilon^2 \mathsf{A}_0}{4} \end{pmatrix} ,
$$

from which it is obvious that its contributions to the mass matrix will be such that they precisely cancel the pole term in (5.9) for $\kappa = 0$. One can easily check that the mass matrix obtained after summing over the three lines is non–vanishing. The total mass matrix however is obtained after summing over *all* the lines in the manifold. and this is not practical with the methods presented here.

5.3. The singlet mass matrix is probably always zero

There is a general argument, rather similar to the one in Ref. [4], to the effect that the superpotential should in fact vanish, and with it the mass matrix for the **1**'s. The present context being somewhat different, we wish to present this argument and also to discuss its possible limitations.

The superpotential in the effective 4-dimensional theory is a complex-analytic section of a holomorphic line-bundle over the space of complex structures. Then, if it is singular, this must happen on a subspace of codimension 1 in the moduli space. Also, the superpotential cannot possibly be singular where the Calabi–Yau manifold is smooth. Therefore, the subset of the moduli space where the superpotential may be singular must be open and dense in the discriminant locus[5]—which we will prove is impossible at least for a great majority of Calabi–Yau manifolds. Finally, being a non-singular analytic section of a non-trivial holomorphic bundle over a compact moduli space, it in fact has to be zero. For technical reasons explained below, we will consider a somewhat redundant but non-singular moduli space.

For the sake of clarity, we first discuss the case of the quintic in $\mathbb{P}_4$ and turn to generalizations below. The effective moduli space of all quintics (singular ones included) is rather badly singular and many of the standard genericity arguments do not apply straightforwardly. For this reason, we shall instead formulate the argument in the (projective) space of coefficients. Since there are 126 quintic monomials on $\mathbb{P}_4$, the projective space of coefficientsis $\mathbb{P}_{125}$; the true (effective) moduli space is obtained by passing to the $PGL(5;\mathbb{C})$ quotient. It is standard that the subspace of singular quintics is of codimension 1 in $\mathbb{P}_{125}$, and its $PGL(5;\mathbb{C})$ quotient is the true discriminant locus (see section 2.2.1 of Ref. [14] for details). Also, the generic point in this 124-dimensional subspace parametrizes a quintic with a single node[6], for which the defining equation must take the following form

$$p(x) = (x_1x_4 - x_2x_3) + p_3(x) + p_4(x) + p_5(x) \ , \tag{5.13}$$

where we chose coordinates so that the node occurs at the point $(0,0,0,0,1)$ in the coordinate patch with $x_5 = 1$, and where the $p_k(x)$ are polynomials of order k in $x_1, \ldots, x_4$.

Now, for a line parametrized by λ which passes through the node $(0,0,0,0,1)$, the local neighborhood of the node may be parametrized as $(\lambda s_1t_1, \lambda s_1t_2, \lambda s_2t_1, \lambda s_2t_2, 1)$. The coordinates $(s_1, s_2; t_1, t_2)$ are homogeneous coordinates of $\mathbb{P}_1\times\mathbb{P}_1$, the local neighborhood of the node projectivised along λ. It is easy to see that the quadratic part of (5.13) vanishes

[5] The discriminant locus is the subspace in the space of complex structures of a complex manifold where the manifold itself is singular.

[6] For an explicit example of a quintic with a single node, see Ref. [23], or section D.3.3 of Ref. [14].

with this parametrization. The remaining terms p_3, p_4, p_5 must vanish separately, as they appear with different powers of λ. Thus, they impose three independent conditions on $\mathbb{P}_1 \times \mathbb{P}_1$, and so have no common solution in general. However, let some $\tilde{s}, \tilde{t}$ be a solution of, say, $p_3 = 0 = p_4$. The remaining equation $p_5(\tilde{s}, \tilde{t}) = 0$ may then be considered as a single condition on the parameters, and so all three p_3, p_4, p_5 vanish on a codimension 1 subset of the 124-dimensional parameter space of singular quintics. This codimension 2 subset of the projective parameter space $\mathbb{P}_{125}$ is then the largest subset of the parameter space over which the corresponding Calabi–Yau manifold contains at least a simple node *and* a line passing through it, and is also the largest subset where the superpotential may possibly diverge. However, this is manifestly neither open nor dense in the codimension 1 discriminant locus, and we conclude that the effective 4-dimensional superpotential cannot diverge. Finally, being an everywhere finite analytic section of a non-trivial holomorphic line-bundle over the projective parameter space $\mathbb{P}_{125}$, the superpotential must in fact vanish. So must also the mass-matrix and all the Yukawa couplings which involve the **1**'s to all orders.

It is possible that this argument does not generalize readily to other known manifolds, and we now discuss to some possible limitations. Firstly, there is the opposite extreme of the above situation—by now familiar from the study of the mirror of $\mathbb{P}_4[5]$ (see Ref. [24]). Here, the space of true (effective) complex structures is 1-dimensional, and the above genericity arguments may be thwarted simply by too low total dimension. Similar limitations will occur also in other cases with low-dimensional moduli spaces.

Recall now that the 1-parameter family of mirrors of $\mathbb{P}_4[5]$ is constructed as a ${\mathbb{Z}_5}^3$-orbifold of a 1-parameter family of ${\mathbb{Z}_5}^3$-symmetric quintics. The difficulties in this case may then also be understood as a consequence of the explicit restriction to highly non-generic quintics. In fact we will work through the case of the mirror quintic below. The above general argument may then appear similarly thwarted in cases where the considered set of Calabi–Yau manifolds is restricted to have special symmetries, and such cases must be re-examined case by case. Whether or not the argument can always be made by deforming into (0,2) theories remains unclear at this time.

Finally, there may well exist Calabi–Yau manifolds for which the discriminant locus is still of codimension 1 in the space of complex structures, but the mildest possible singularization is worse than a conifold with a single node. For example, if it should happen that the mildest singularization includes a singular curve, the world sheet instanton might map the entire world sheet into the singular curve, and our above argument would fail. While we were not able to find an explicit counter-example of this kind, it does not seem impossible that manifolds embedded in weighted projective spaces might exhibit this phenomenon owing to special (in)divisibility properties of the weights. Also, examples of Calabi–Yau manifolds exist (see section 3.5.1 of Ref. [14]) which are defined by overdetermined systems of equations, and about which far too little is known to make any definite claims.

However, one general comment is in order for these latter cases, and in fact for all possible candidate counter-examples. Namely, it is well known that not all deformations of the complex structure of an abstract Calabi–Yau manifold need be representable as

deformations of a given, concrete embedding. It is then perfectly possible that the considered embedding is constrained in just such a way that the mildest singularization of the embedding develops a singular curve, rather than an isolated node. This by itself would *not* constitute a counter-example to the above general argument, as long as a deformation of the abstract manifold exists which smoothes the singular curve(s) into one or more isolated nodes. Unfortunately, effects of such (abstract) deformations of the manifold are much harder to study than the explicitly realizable (polynomial) embedding deformations. Therefore, the rigorously proven *full* scope of the above claim, that all the terms in the superpotential which involves should vanish, remains an open question.

Silverstein and Witten [4] have stressed the role of the mass matrix as an obstruction to the possibility of deforming a $(2,2)$-theory to a $(0,2)$ theory. We have argued above that, in simple cases, the total mass is zero since for a generic conifold the instantons do not pass through the node. For particular manifolds however this may not be true. An example of this is the mirror of $\mathbb{P}_4[5]$. This manifold [25], which we shall call $\mathcal{W}$, can be realized by identifying the points of a covering manifold $\widehat{\mathcal{W}}$ that corresponds to the quintic

$$p \stackrel{\text{def}}{=} X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 - 5\psi\, X_1X_2X_3X_4X_5 = 0 \tag{5.14}$$

in $\mathbb{P}_4[5]$ under the action of a group which is abstractly $\mathbb{Z}_5^3$ and which has the generators

$$\begin{aligned}&(\mathbb{Z}_5;\ 1,0,0,0,4)\\ &(\mathbb{Z}_5;\ 0,1,0,0,4)\\ &(\mathbb{Z}_5;\ 0,0,1,0,4)\ .\end{aligned}$$

The notation indicates that the first generator, for example, has the action

$$(X_1, X_2, X_3, X_4, X_5) \to (\alpha X_1, X_2, X_3, X_4, \alpha^4 X_5)\ ; \qquad \alpha = e^{\frac{2\pi i}{5}}\ .$$

Now the polynomial (5.14) is constrained to have only one parameter by the symmetries. The manifold $\widehat{\mathcal{W}}$ becomes singular when $\psi^5 = 1$. The resulting variety was studied by Schoen [26] because of its special properties before its importance for mirror symmetry was realized. Before taking the quotient by the group the variety $\widehat{\mathcal{W}}$ has 125 nodes at the points

$$(\alpha^{n_1}, \alpha^{n_2}, \alpha^{n_3}, \alpha^{n_4}, \alpha^{n_5}) \qquad \text{with} \qquad \sum_i \alpha^{n_i} = 0\ .$$

These points are all identified under the group so that $\mathcal{W}$ has only one node. All the nodes are equivalent in virtue of the symmetries so consider for example the node $(1,1,1,1,1)$. Schoen has observed that the 24 lines that connect this node to the others whose coordinates are the permutations of $(\alpha, \alpha^2, \alpha^3, \alpha^4, 1)$ actually lie in the hypersurface $p = 0$. The group identifies the nodes and the lines so that on $\mathcal{W}$ we not only have one line that passes through the node but the line also has self intersection at the node!

In this situation we can have recourse to an argument of Silverstein and Witten and deform the theory into a $(0,2)$ theory. In the context of the linear sigma model [27] the instanton contribution can only be singular if an instanton passes through a point $X^\sharp$ for which

$$\frac{\partial p}{\partial X_i} + q_i = 0 \tag{5.15}$$

with q_i a vector of quartics corresponding to a singlet deformation. It now suffices merely to show that no instanton passes through the point $X^\sharp$ specified by this equation. On $\mathcal{W}$ there are four quartic forms that are not derivatives of quintics. Let

$$r \;=\; X_1X_2X_3X_4X_5 \qquad \text{and} \qquad r_i \;=\; \frac{r}{X_i}$$

so that r_i is the product of the X_j with X_i deleted. There are five forms $r_i\, dX_i$ that are invariant under the group but there is a relation

$$\sum_i r_i\, dX_i \;=\; dr$$

so only four are independent. This being so we take for the singlet deformation

$$q \;=\; 5\epsilon \sum_{i=1}^{5} \lambda_i r_i dX_i \qquad \text{with} \qquad \sum_{i=1}^{5} \lambda_i \;=\; 0\ .$$

The equations $p_{,i} + q_i = 0$ can be solved only when

$$\prod_{i=1}^{5} (\psi - \epsilon\lambda_i) \;=\; 1 \qquad \text{and} \qquad X_i \;=\; (\psi - \epsilon\lambda_i)^{\frac{1}{5}}\ ,$$

where, by means of the scaling symmetry, we have set $r = 1$ (note that r cannot be zero or all the X_i's would vanish too). We can, for simplicity, solve these equations perturbatively in ϵ. We find

$$\psi \;=\; 1 - \epsilon^2 \sum_{i>j} \lambda_i\lambda_j \qquad \text{and} \qquad X_i \;=\; 1 - \frac{\epsilon}{5}\lambda_i\ .$$

If we impose the constraint

$$\sum_i \lambda_i^2 \;=\; 0 \qquad \text{in addition to} \qquad \sum_i \lambda_i \;=\; 0$$

then we fix $\psi = 1$ and we are able to move $X^\sharp$ in a three dimensional neighborhood of the node. Since there are points in such a neighborhood through which lines do not pass it is clear that the one-instanton contribution to the mass cannot be singular and hence must vanish. The conclusion is that the total mass and the Yukawa couplings that involve **1**'s vanish in this case. Note however that the argument turned out to be rather delicate. A self intersecting line always passes through the node and of the 224 **1**'s (there must be 224 by mirror symmetry) only four can be represented by polynomials. These four however are sufficient to complete the argument.

6. **Yukawa Couplings**

We now turn to evaluating the $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ and then $\mathbf{1}^3$ couplings using the general formula (1.1).

6.1. The coupling in terms of zero modes

We have chosen local coordinates X^1, X^2 and X^3 so that $X^3 = \zeta$ is aligned with the instanton as in (3.4). The instantino modes are given in Eq. (3.5). Note that we have used the components of the metric tensor to lower the indices; in fact, that is how they will naturally occur in the vertex operators in Eq. (3.6)–(3.11).

Before turning to the full calculation, notice that the couplings that interest us have a common factor. Consider first the integral over the $\bar{\psi}$'s. Since the "0-picture" of the vertex operators is obtained by replacing the $\mathrm{e}^{-\phi(\overline{z})}\,\psi^{\overline{\mu}}(\overline{z})$ field combination with $\Omega^{\overline{\mu}}{}_{\nu\rho}\,\psi^{\nu}(\overline{z})\,\psi^{\rho}(\overline{z})$ and since there are two $\psi^{\bar{3}}$-instantinos, we always obtain the factor

$$\int \mathrm{d}\overline{\delta}\,\mathrm{d}\overline{\gamma}\;\psi^{\bar{\zeta}}(\overline{z})\psi^{\bar{\zeta}}(\overline{w}) \;=\; \frac{(\overline{z}-\overline{w})}{(\overline{cz}+\overline{d})^2\,(\overline{cw}+\overline{d})^2}\;,$$

which, as in our discussion of the mass matrix, again cancels the factor $(\overline{z}-\overline{w})^{-1}$ resulting from the free correlator $\big\langle\,\mathrm{e}^{-\phi(\overline{z})}\,\mathrm{e}^{-\phi(\overline{w})}\,\big\rangle$. Moreover, the two $\bar{\psi}$-instantinos necessarily occur on different vertices, while $\psi\,\psi$ occurs on the third vertex, so no selection rule can ever come from the ψ-integration.

On the other hand, the λ's and the $\bar{\lambda}$'s are somewhat more "mobile" and, in fact, the selection rule of Ref. [7] came precisely from the fact that the two modes of λ had to sit on the same vertex; indeed, if $z = w$, Eq. (4.3) yields identically zero.

Now consider the world–sheet instanton correction to the $\mathbf{1}^3$ coupling. It will necessarily contain

$$[\,\lambda\,\bar{\lambda}\,]_{(z_1)}\,[\,\lambda\,\bar{\lambda}\,]_{(z_2)}\,[\,\lambda\,\bar{\lambda}\,]_{(z_3)}\;.$$

When substituting the zero-modes of the λ's and $\bar{\lambda}$'s, at least one λ and at least one $\bar{\lambda}$ has to be repeated and the Berezin integration yields zero. This fate may be be evaded only if we contract a $\lambda^{\alpha}(z)$ with a $\lambda_{\beta}(w)$ (from different vertices). This can be done in several different ways, each bringing down a $\delta^{\alpha}_{\beta}(X)/(z-w)$ factor. Had we not lowered the indices on $\lambda^{\bar{\xi}}$ and $\lambda^{\bar{\eta}}$, there would be factors of $g^{\alpha\overline{\beta}}$ instead of δ^{α}_{β}.

Consider now the world–sheet instanton correction to the mixed couplings. We can readily evaluate both the $\Big\langle\,V^{I}_{(-1)}\,V^{\bar{J}}_{(-1)}\,V^{1}_{(0)}\,\Big\rangle$ and the $\Big\langle\,V^{0}_{(-1)}\,V^{\overline{0}}_{(-1)}\,V^{1}_{(0)}\,\Big\rangle$ coupling. In the latter one, the λ's and the $\bar{\lambda}$'s are distributed as follows

$$[\,\lambda\,\lambda\,]_{(z_1)}\,[\,\bar{\lambda}\,\bar{\lambda}\,]_{(z_2)}\,[\,\lambda\,\bar{\lambda}\,]_{(z_3)}\;.$$

It suffers from the same abundance in λ's and $\bar{\lambda}$'s as the instanton correction to the $\mathbf{1}^3$ one, so it must vanish, unless we contract a λ from the first vertex with any of the $\bar{\lambda}$'s. (Contracting the λ in the third, E_6–$\mathbf{1}$ vertex does not help since the remaining two λ's are both λ^3 and yield zero if they occur at the same vertex.)

Let us now consider the various contributions to the correlation functions. The $\mathbf{10}.\overline{\mathbf{10}}.\mathbf{1}$ part of the $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ coupling is proportional to

$$[\lambda^I\,\bar{\lambda}]_{(z_1)}\,[\lambda^{\bar{J}}\,\lambda]_{(z_2)}\,[\lambda\,\bar{\lambda}]_{(z_3)} \;=\frac{\delta^{I\bar{J}}}{(z_1-z_2)}[\bar{\lambda}]_{(z_1)}\,[\lambda]_{(z_2)}\,[\lambda\,\bar{\lambda}]_{(z_3)}\ .$$

We may choose the 0-picture for either of the three vertices, so there are really three such couplings. A straightforward substitution gives

$$\begin{aligned}\Big\langle\, V^I_{(-1)}\,V^{\bar{J}}_{(-1)}\,V^1_{(0)}\,\Big\rangle &= \int \mathrm{d}^6 z\ \frac{-1}{(\overline{z}_1-\overline{z}_2)}\,\frac{+\delta^{I\bar{J}}}{(z_1-z_2)} h_{\overline{\mu}}{}^{\alpha}\, b_{\beta\overline{\nu}}\,\Omega^{\overline{\rho}\,\overline{\sigma}\,\overline{\tau}}\ s_{\overline{\rho}}{}^{\gamma}{}_{\delta}\\ &\times\int \mathrm{d}^4\overline{\alpha}\,\Big[\, -\psi^{\overline{\mu}}_{(1)}\,\psi^{\overline{\nu}}_{(2)}\,\psi^{(3)}_{\overline{\sigma}}\,\psi^{(3)}_{\overline{\tau}}\,\Big]\\ &\times\int \mathrm{d}^4\alpha\,\Big[\,\lambda^{\beta}_{(2)}\,\lambda^{\delta}_{(3)}\,\lambda^{(1)}_{\alpha}\,\lambda^{(3)}_{\gamma}\,\Big]\ ,\end{aligned}$$

where the signs come from permuting the various fermions. The parenthetical indices remind us from which vertex the indexed field came and, hence, on which z_i it depends. Thus we have:

$$\begin{aligned}\Big\langle\, V^I_{(-1)}\,V^{\bar{J}}_{(-1)}\,V^1_{(0)}\,\Big\rangle &= 2\delta^{I\bar{J}}\int\frac{\mathrm{d}^2 z_2}{|cz_2+d|^4}\ b_{3\bar{3}}(X(z_2))\\ \times\ \int\frac{\mathrm{d}^2 z_1\,\mathrm{d}^2 z_3}{|cz_1+d|^4|cz_3+d|^4}&\ \Big(\frac{cz_1+d}{cz_3+d}\Big)\Big(\frac{z_2-z_3}{z_1-z_2}\Big) h_{\bar{3}}{}^{[1}(X(z_1))\, s_{\bar{3}}{}^{2]}{}_3(X(z_3))\ ,\end{aligned}\tag{6.1}$$

where $h^{[1}\,s^{2]}\stackrel{\rm def}{=}h^1 s^2-h^2 s^1$. The other two expressions,

$$\Big\langle\, V^I_{(-1)}\,V^{\bar{J}}_{(0)}\,V^1_{(-1)}\Big\rangle\qquad\text{and}\qquad\Big\langle\, V^I_{(0)}\,V^{\bar{J}}_{(-1)}\,V^1_{(-1)}\Big\rangle\ ,$$

yield the same result as in Eq. (6.1), as do the mixed cubic Yukawa couplings with all the other $SO(10)\subset E_6$ components of the $\mathbf{27}$'s and $\overline{\mathbf{27}}$'s.

Consider now the $\mathbf{1}^3$ coupling:

$$\begin{aligned}\Big\langle\, V^1_{(-1)}\,V^1_{(-1)}\,V^1_{(0)}\,\Big\rangle &= \int \mathrm{d}^6 z\ \frac{1}{(\overline{z}_2-\overline{z}_1)}\ s_{\overline{\mu}}{}^{\alpha}{}_{\beta}\ s_{\overline{\nu}}{}^{\gamma}{}_{\delta}\ \Omega^{\overline{\rho}\,\overline{\sigma}\,\overline{\tau}}\ s_{\overline{\rho}}{}^{\eta}{}_{\kappa}\\ &\times\int \mathrm{d}^4\overline{\alpha}\,\Big[\,+\psi^{\overline{\mu}}_{(1)}\,\psi^{\overline{\nu}}_{(2)}\,\psi^{(3)}_{\overline{\sigma}}\,\psi^{(3)}_{\overline{\tau}}\,\Big]\\ &\times\int \mathrm{d}^4\alpha\,\Big[\,-\lambda^{\beta}_{(1)}\,\lambda^{\delta}_{(2)}\,\lambda^{\kappa}_{(3)}\,\lambda^{(1)}_{\alpha}\,\lambda^{(2)}_{\gamma}\,\lambda^{(3)}_{\eta}\,\Big]^!\ .\end{aligned}$$

Since the three $\lambda_{(i)}$'s are already on different vertices, we can contract any one of the $\lambda_{(i)}$'s with any of the $\lambda^{(i)}$'s from either of the other two vertices. (Contracting λ's from the same vertex would yield a pole of first order, but such contributions do not arise as the $s_{\overline{\mu}}{}^{\alpha}{}_{\beta}$'s are traceless.) The remaining six contributions are of the form of

$$-\left\langle\, \lambda^{\beta}_{(1)}\, \lambda^{(2)}_{\gamma} \,\right\rangle \int \mathrm{d}^4\alpha\; \lambda^{\delta}_{(2)}\, \lambda^{\kappa}_{(3)}\, \lambda^{(1)}_{\delta}\, \lambda^{(3)}_{\eta}\ ,$$

which evaluates to

$$-\frac{\delta^{\beta}_{\gamma}\, \delta^{\delta}_{3}\, \delta^{\kappa}_{3}\, (\delta^1_{\alpha}\, \delta^2_{\eta} - \delta^2_{\alpha}\, \delta^1_{\eta})}{(\overline{cz}_1 + \overline{d})^2\, (\overline{cz}_2 + \overline{d})^2\, (\overline{cz}_3 + \overline{d})^2} \left(\frac{z_2 - z_3}{z_1 - z_2}\right)\left(\frac{cz_1 + d}{cz_3 + d}\right) .$$

Collecting all contributions, we get the simple answer

$$\begin{aligned}\left\langle\, V^1_{(-1)}\, V^1_{(-1)}\, V^1_{(0)} \,\right\rangle = \int \frac{\mathrm{d}^2 z_1}{|cz_1 + d|^4}\, \frac{\mathrm{d}^2 z_2}{|cz_2 + d|^4}\, \frac{\mathrm{d}^2 z_3}{|cz_3 + d|^4} \left(\frac{cz_1 + d}{cz_3 + d}\right)\left(\frac{z_2 - z_3}{z_1 - z_2}\right) \\ \times\, s_{\bar{3}}{}^{[1|}{}_{\alpha}(X(z_1))\; s_{\bar{3}}{}^{\alpha}{}_{3}(X(z_2))\; s_{\bar{3}}{}^{|2]}{}_{3}(X(z_3))\ .\end{aligned} \tag{6.2}$$

Compare now the results in Eq. (6.1) and (6.2). On inverting Eq. (3.4),

$$z_i \;=\; -\frac{d\zeta_i - b}{c\zeta_i - a}\ ,$$

we see that

$$\left(\frac{cz_1 + d}{cz_3 + d}\right)\left(\frac{z_2 - z_3}{z_1 - z_2}\right) \;=\; \left(\frac{\zeta_2 - \zeta_3}{\zeta_1 - \zeta_2}\right)\ ,$$

so that, for example,

$$\left\langle\, V^1_{(-1)}\, V^1_{(-1)}\, V^1_{(0)} \,\right\rangle = \int \mathrm{d}^2\zeta_1\, \mathrm{d}^2\zeta_2\, \mathrm{d}^2\zeta_3 \left(\frac{\zeta_2 - \zeta_3}{\zeta_1 - \zeta_2}\right) s_{\bar{\zeta}}{}^{[\xi|}{}_{\alpha}(\zeta_1)\; s_{\bar{\zeta}}{}^{\alpha}{}_{\zeta}(\zeta_2)\; s_{\bar{\zeta}}{}^{|\eta]}{}_{\zeta}(\zeta_3)\ .$$

Therefore, up to the overall factor, all three results (6.1) and (6.2), and all other components of the $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ and $\mathbf{1}^3$ couplings are of the form

$$\int \mathrm{d}^2\zeta_1\, \mathrm{d}^2\zeta_2\, \mathrm{d}^2\zeta_3 \left(\frac{\zeta_2 - \zeta_3}{\zeta_1 - \zeta_2}\right) A(\zeta_1)\, B(\zeta_2)\, C(\zeta_3)\ . \tag{6.3}$$

We turn now to a discussion of the form of these expressions.

6.2. The form of the Yukawa couplings

The expression we obtain for the $\mathbf{1}^3$ coupling is proportional to a sum of terms of the form

$$\kappa(a,b,c) \stackrel{\text{def}}{=} \frac{1}{2}\varepsilon_{ij}\int d^2\zeta_1 d^2\zeta_2 d^2\zeta_3 \left(\frac{\zeta_2-\zeta_3}{\zeta_1-\zeta_2}\right) a_{\bar{\zeta}}{}^{i}{}_{\mu}(\zeta_1) b_{\bar{\zeta}}{}^{\mu}{}_{\zeta}(\zeta_2) c_{\bar{\zeta}}{}^{j}{}_{\zeta}(\zeta_3) \qquad (6.4)$$

where the indices i and j run over the transverse coordinates ξ and η. The $b_{\bar{\zeta}}{}^{\mu}{}_{\zeta}$ here should not be confused with the (1,1)-form $b_{\mu\bar{\nu}}$.

We make use also of the full form of Cauchy's Theorem:

$$\int_{\partial B}\frac{d\zeta}{\zeta}\varphi = 2\pi i\varphi(0) - \int_B \frac{d\zeta}{\zeta}\bar{\partial}\varphi \qquad (6.5)$$

which is a consequence of the identity

$$\bar{\partial}\left(\frac{d\zeta}{\zeta}\right) = -\pi\delta^{(2)}(\zeta)\, d\zeta d\bar{\zeta}\ ,$$

where $\delta^{(2)}(\zeta)$ means $\delta(\Re e\,\zeta)\delta(\Im m\,\zeta)$, and Stoke's Theorem. Cauchy's Theorem is of course most often applied to holomorphic functions in which case the integral on the right hand side of (6.5) vanishes. For our present purpose we wish to apply (6.5) on the world–sheet[2], Σ, so we take B to be a disk of radius R and let $R\to\infty$. If we assume also that φ approaches a finite limit at ∞ then the boundary contribution becomes $2\pi i\varphi(\infty)$ and hence

$$\int_\Sigma \frac{d\zeta}{\zeta}\bar{\partial}\varphi = 2\pi i(\varphi(0)-\varphi(\infty))\ .$$

Proceeding in this manner we find that the integral in (6.4) is proportional to

$$\varepsilon_{ij}\int_{\Sigma_1\times\Sigma_2}(a\times b)^i(\zeta_1)\, c^j(\zeta_2)\,(\zeta_1-\zeta_2) \qquad (6.6)$$

with

$$(a\times b)^j = a^j\beta - \alpha^j{}_k b^k \quad , \quad a^i = a_{\bar{\zeta}}{}^{i}{}_{\zeta} d\zeta d\bar{\zeta} \quad , \quad b^i = b_{\bar{\zeta}}{}^{i}{}_{\zeta} d\zeta d\bar{\zeta} \quad , \quad c^i = c_{\bar{\zeta}}{}^{i}{}_{\zeta} d\zeta d\bar{\zeta}\ .$$

Note that the above expression (6.6) is non-zero because the surface term in (6.5) does not vanish. This phenomenon is related to the existence of unphysical particles whose contribution to the path integral vanishes as it takes the form of a total derivative. However, when the surface term, as above, does not vanish they become physical. In order

[2] or rather to its image in the Calabi–Yau manifold, the rational curve, which is also an $S^2 = \mathbb{P}_1$

to remedy this and guarantee unitarity one needs to take the contribution from contact terms into account in order to cancel the surface terms [28]. Let us now turn to this in more detail.

The Yukawa coupling that we have calculated has the unattractive feature that it seems not to be well-defined on BRST cohomology classes, that is, it seems to depend on the precise representatives chosen for the cohomology classes in $\mathrm{H}^1(\mathrm{End}\mathcal{T})$. On the other hand, it is also given by a sum of δ-function contributions where two of the three vertex operators coincide. This latter property is the key to "curing" the above ambiguity.

Recall that, in general, the correlation functions of vertex operators are ambiguous up to the addition of contact terms when vertex operators collide. Generally, one can fix these contact terms by demanding that the Ward identities of the theory be preserved. In the above calculation, we blithely assumed that the contact terms vanished. We found, unfortunately, that the resulting amplitude was not well-defined on BRST cohomology classes. Clearly then, *zero* is the wrong choice for the contact terms. The correct choice – the one which renders the amplitude well-defined on BRST cohomology classes – is to add to the amplitude a contact contribution which precisely cancels (6.6).

There are several things to be noted about the proposed contact terms.

- The ambiguous amplitude (6.6) had no "bulk" contribution. It was given entirely by a contact interaction. It is only in this felicitous circumstance that one can cancel it by a choice of contact terms.
- The contact term, for a given representative $a^{\mu}{}_{\nu}$, depends on the restriction of $a^{i}{}_{j}$ and $a^{\zeta}{}_{\zeta}$ to the curve. For any given curve, there always exists a choice of representative such that these vanish (and hence so does the contact term) when restricted to that curve.
- The mass term of Sections 4 and 5 depends on the restriction of $a^{i}{}_{\zeta}$ to the curve. Indeed, it depends only on the cohomology class of $a^{i}{}_{\zeta}$. It is unambiguously-defined, and receives no contribution from the contact term discussed here.

It would be desirable to be able to write a universal expression for the contact term which reduces to the right thing when restricted to any given curve, but we have not been able to do so.

Finally, consider the $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ coupling. The expression (6.1), reduces to (omitting the E_6 indices)

$$\kappa(h,b,s) = \frac{1}{2}\varepsilon_{ij}\int \mathrm{d}^6\zeta \left(\frac{\zeta_2-\zeta_3}{\zeta_1-\zeta_2}\right)\; h_{\bar\zeta}{}^{i}(\zeta_1)\, b_{\zeta\bar\zeta}(\zeta_2)\, s_{\bar\zeta}{}^{j}{}_{\zeta}(\zeta_3)\ .$$

We may set

$$b = b_{\zeta\bar\zeta} d\zeta d\bar\zeta\ ,\ \ s^j = s_{\bar\zeta}{}^{j}{}_{\zeta} d\zeta d\bar\zeta\ ,\ \text{and}\ h_{\bar\zeta}{}^{i} d\bar\zeta = \bar\partial\chi^i$$

and integrate by parts to find

$$\kappa(h,b,s) = \frac{1}{2}\varepsilon_{ij}\int_{\Sigma_1\times\Sigma_2} (\zeta_1-\zeta_2)\,\chi^i(\zeta_1)\, b(\zeta_1)\, s^j(\zeta_2)\ .$$

As for $\mathbf{1}^3$, the surface terms is canceled by taking the contact terms into account. Thus, we find that to first order in the instanton expansion both $\mathbf{1}^3$ and $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ vanish. This is in perfect agreement with the general argument of Section 5.3–that all Yukawa couplings involving $\mathbf{1}$'s vanish to all orders. Though it would clearly be preferable to give a clearer account of the rôle of the contact terms.

7. **Conclusion**

7.1. Summary and discussion

Complementing the extensive and rather complete understanding of the massless **27**'s and $\overline{\mathbf{27}}$'s and their Yukawa couplings in a Calabi–Yau compactification, we have examined the situation with the **1**'s.

Some striking differences emerge already at the classical level: foremost, that the number of **1**'s is *not* constant when the complex structure is varied, but jumps at special R-symmetric subregions; see Section 2.2. By mirror symmetry, the same is expected with respect to variations of the Kähler class. Unfortunately, this means that the total space of parameters associated to the **27**'s, $\overline{\mathbf{27}}$'s and **1**'s is not as uniform as the well studied $\mathbf{27}+\overline{\mathbf{27}}$'s subspace. Nevertheless, for the generic part of this total moduli space where the number of **1**'s is constant, the $\mathbf{1}^3$ and $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ couplings can be shown to vanish among the **1**'s which have a deformation-theoretic representation. Thus, at least these **1** parameters represent integrable deformations.

We then turn to the first order instanton corrections, which depend on the families of lines (genus-0 curves) of degree 1 in the Calabi–Yau space $\mathcal{M}$. A generic, $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ line is found to produce a nonvanishing entry in the mass matrix for the **1**'s. However, as the complex structure of $\mathcal{M}$ is varied, these lines move about and their contributions were seen to be finite (though not zero) in a simple case where two or more lines in $\mathcal{M}$ coalesce into a single line. However, we present general arguments that the sum over all instantons (at any given order) vanishes.

The first order instanton corrections to the $\mathbf{1}^3$ and $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ Yukawa couplings were all found to be of the same generic form. Moreover, these corrections can (and should) be eliminated through a judicious inclusion of contact terms. This has the virtue of preserving not only unitarity, but also the vanishing of (some of the) $\mathbf{1}^3$ and $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ Yukawa couplings and so also the integrability of the **1**-deformations found at the classical level.

Finally, following [4], in Section 5.3 we give a general argument that all terms in the superpotential involving the **1**'s vanish to all orders. There we also discuss the possible limitations of this argument.

7.2. Open questions

Some of the open questions raised by the considerations of this article are:

- We do not consider the expressions that we have found for the instanton contributions to the $\mathbf{1}^3$ and $\mathbf{27}.\overline{\mathbf{27}}.\mathbf{1}$ couplings to be satisfactory. The reader may care to set this matter in order.

- The present discussions of the mass-matrix and the Yukawa couplings leaves untouched the difficult and important issue of the kinetic term for the **1**'s. That is, we need a better understanding of the geometry of the space of singlets, *i.e.* the analogue of special geometry.

- It remains unclear how to sum up explicitly the contributions of all the lines to the mass-matrix even for a simple model such as $\mathbb{P}_4[5]$.

- A technical point: the expression (2.4) that we use to relate polynomial-forms $\mathsf{a}_M dx^M$ to elements of $H^1(\mathcal{M}, \mathrm{End}\,\mathcal{T})$ is in reality an ansatz. It would be nice to have a derivation of this via a residue calculation, say, as in [9] for the polynomial deformations of $H^1(\mathcal{M}, \mathcal{T})$.

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A. Remarks on the Determinants and Resolution of an Apparent Paradox

It is instructive to inquire how the individual factors in μ vary with ϵ. We first choose a basis of zero modes

$$\zeta_r = \frac{\partial\zeta}{\partial a^r}\ , \qquad \psi_\rho = \frac{\partial\psi}{\partial\alpha^\rho}\ , \qquad \lambda_{\bar\sigma} = \frac{\partial\lambda}{\partial\alpha^{\bar\sigma}}\ , \tag{A.1}$$

and set

$$G^B_{r\bar s} = \int \frac{d^2z}{\nu^2}\, g_{\zeta\bar\zeta}\,\zeta_r\bar\zeta_{\bar s}\ , \qquad G^F_{\rho\bar\sigma} = \int \frac{d^2z}{\nu}\, g(\psi_\rho, \lambda_{\bar\sigma})\ , \tag{A.2}$$

(the single factor of ν in the second metric is due to the inclusion of a factor of $(h^{zz})^{\frac{1}{2}}$ that takes account of the fact that the fermionic zero modes are spinors). Although this is not immediately apparent from (A.2) it is straightforward to check that G^B and G^F are in fact independent of the parameters (a, b, c, d) of the instantons. The bosonic metric, $\det(G^B)$ has a limit as $\epsilon \to 0$ that is finite and non zero. The behaviour of the other factors is however more interesting. By taking the metric g that appears in (A.2) to be the restriction on the Fubini–Study metric to the Calabi–Yau manifold, it is straightforward to compute the dependence of $G^F_{\rho\bar\sigma}$ on ϵ. Consider for example the case $\kappa = 0$ for the lines l_ϵ of Section 5.2. We find that for such a line

$$\det G^F \approx |\epsilon|^{-4}\ .$$

(The symbol $\approx$ means asymptotic equality up to multiplication by a constant.) Since we have argued that μ is constant it must therefore be the case that

$$\frac{\mathrm{Det}'(\bar\partial^\dagger_{\mathcal{T}\otimes\mathcal{S}}\bar\partial_{\mathcal{T}\otimes\mathcal{S}})}{\mathrm{Det}'(\bar\partial^\dagger_{\mathcal{T}}\bar\partial_{\mathcal{T}})} \approx |\epsilon|^{-4}\ . \tag{A.3}$$

From (A.3) we see that the basis (3.5) that we have been using for the fermionic zero modes is singular as $\epsilon \to 0$. It is curious that this is the best choice of basis (since for this basis μ is constant). Consider then the following basis, which might seem a better choice ($\psi^{\bar\zeta}$ and λ^{ζ} are as before)

$$\begin{aligned}
\psi_{\bar\xi} &= -\frac{\overline{\alpha}\bar\epsilon^2}{\bar c\bar z + \bar d} & \qquad \lambda_\xi &= -\frac{\alpha\epsilon^2}{cz + d} \\
\psi_{\bar\eta} &= \frac{\overline{\alpha}\bar a}{\bar c(\bar c\bar z + \bar d)} + \frac{\overline{\beta}}{\bar c\bar z + \bar d} & \qquad \lambda_\eta &= \frac{\alpha a}{c(cz + d)} + \frac{\beta}{cz + d}
\end{aligned} \tag{A.4}$$

Now given the relation

$$\begin{aligned} \xi - \frac{\zeta\eta}{\epsilon^2} &\to \xi_0 \\ \eta &\to \eta_0 \\ \zeta &\to \zeta_0 \ . \end{aligned} \tag{A.5}$$

between the coordinates (ξ, η), for the line $l_{+\epsilon}$ of type $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and (ξ_0, η_0), for the line of type $\mathcal{O}(0) \oplus \mathcal{O}(-2)$, we find, to leading order,

$$\begin{aligned} \frac{\partial}{\partial \xi_0} &\sim \frac{\partial}{\partial \xi} \ , \\ \frac{\partial}{\partial \eta_0} &\sim \frac{\zeta}{\epsilon^2}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \ . \end{aligned}$$

It follows that

$$\begin{aligned} \psi_{\bar{\xi}_0} &\sim \psi_{\bar{\xi}} \sim -\frac{\overline{\alpha}\epsilon^2}{\bar{c}\bar{z} + \bar{d}} \to 0 \ , \\ \psi_{\bar{\eta}_0} &\sim \frac{\zeta}{\epsilon^2}\psi_{\bar{\xi}} + \psi_{\bar{\eta}} \to \frac{\overline{\alpha}}{\bar{c}(\bar{c}\bar{z} + \bar{d})^2} + \frac{\overline{\beta}}{\bar{c}\bar{z} + \bar{d}} \ . \end{aligned}$$

In this basis $\det G^F$ has a finite and nonzero limit as $\epsilon \to 0$. So now $\mu \approx |\epsilon|^{-4}$. However, the reader may easily repeat the steps in the computation of the mass matrix for the new basis. The only change is the replacement $m \to |\epsilon|^4 m$ so the product $m\mu$ is unchanged as it must be.

We learn that the pole term in the mass matrix owes its existence to an eigenvalue of the bosonic operator which vanishes as $\epsilon \to 0$. This in turn we understand from the fact that $h^0(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) = 0$ but $h^0(\mathcal{O}(0) \oplus \mathcal{O}(-2)) = 1$ from which we see that the bosonic operator acquires a new zero mode precisely at $\epsilon = 0$. In fact we may understand in the same way why an $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ line does not give rise to a pole. For an $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ line there are two new zero modes for the bosonic operator but also two new zero modes for the fermionic operator. These cancel so μ has no pole. The zero mode calculation gives a factor that tends to zero. Thus an $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ line gives a vanishing contribution to the mass matrix as anticipated in 4.2. It is interesting that the order of the degeneration of the instanton metric is related to the number of zero modes of the operators appearing in (A.3).

We can now dispose rather easily of an apparent paradox that appears when one tries to calculate the mass matrix directly for an $\mathcal{O}(0) \oplus \mathcal{O}(-2)$ line. In this case, one would choose fermion zero modes

$$\begin{aligned} \psi_{\bar{\xi}_0} &= 0 \ , & \lambda_{\xi_0} &= 0 \ , \\ \psi_{\bar{\eta}_0} &= \frac{\overline{\alpha}}{\bar{c}(\bar{c}\bar{z} + \bar{d})^2} + \frac{\overline{\beta}}{\bar{c}\bar{z} + \bar{d}} & \lambda_{\eta_0} &= \frac{\alpha}{c(cz + d)^2} + \frac{\beta}{cz + d} \ . \end{aligned} \tag{A.6}$$

and one finds by repeating the steps of §4.1 that the contribution of the zero modes vanishes. So it is tempting to conclude that the contribution of such a line to the mass matrix is zero. This, however, is a trap for the unwary. If instead of setting $\epsilon = 0$ from the outset we let $\epsilon \to 0$ and employ the basis (A.4) then, as we have seen above, the zero mode calculation gives a contribution

$$m^0 \approx \frac{|\epsilon|^4}{\epsilon^2} = \bar{\epsilon}^2 \ ,$$

which indeed tends to zero with ϵ. The contribution of the nonzero modes, however, is $1/|\epsilon|^4$ so the asymptotic behaviour is in fact $1/\epsilon^2$.

It is instructive also to consider now the $\overline{\mathbf{27}}^3$ coupling from the same perspective. The same apparent paradox arises if one tries to calculate the contribution of an $\mathcal{O}(0) \oplus \mathcal{O}(-2)$ line by integrating directly over the double line. The zero modes (A.6) are such that the zero mode integral vanishes. However we know that the contribution of a double line to the $\overline{\mathbf{27}}^3$ coupling is not zero. It is twice the contribution of a single line; and all the single lines contribute an equal and non-zero amount. The resolution is as before: if we work with the moving basis (A.4) then, as is easily seen, the zero mode integral is proportional to $|\epsilon|^4$ while $\mu \approx |\epsilon|^{-4}$, due as before to the fact that one of the eigenvalues of the bosonic operator tends to zero as $|\epsilon|^4$. In fact by turning these considerations around we could have seen sooner that the bosonic operator must have an eigenvalue that vanishes as $|\epsilon|^4$ and from this the $1/\epsilon^2$ pole in the mass term follows easily and inevitably.

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